An overview of the analytic computation of multi-loop Feynman integrals

Yang Zhang 张扬

University of Science and Technology of China

Department of Modern Physics, USTC

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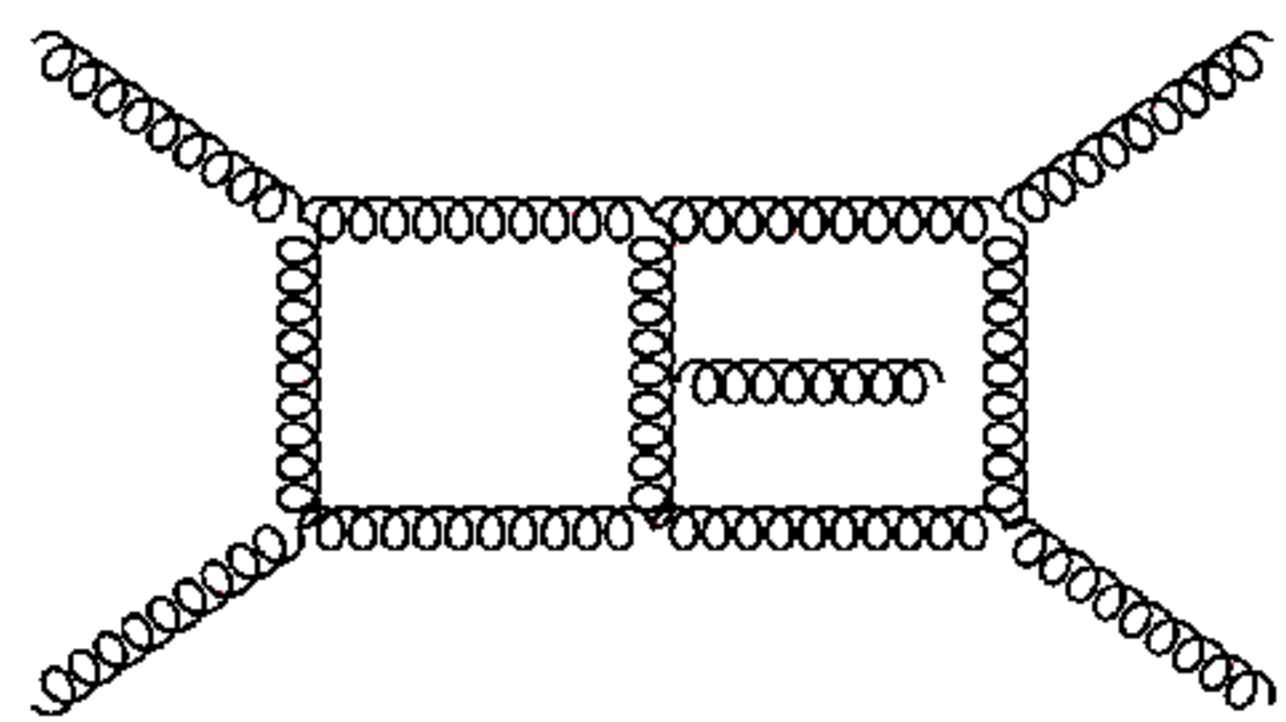
Disclaimer

I try to be pedagogical in this seminar talk…

Why analytic Feynman integrals?

- Once the analytic expression is obtained, the phase point generation is extremely fast
- Avoid unstable numeric phase points
- Understand the deep structure and hidden symmetry in quantum field theory

- and yes, we can.



The State of Art

recent analytic results

- 2-loop qqbar → ttbar nonplanar integrals
  (Di Vita, Gehrmann, Laporta, Mastrolia, Pierpaolo; Primo, Schubert, Ulrich, 2019)
- 4-loop form factor planar integrals
  (von Manteuffel, Schabinger, 2019)
- 2-loop Higgs + one jet production (with t quark mass dep.) nonplanar integrals (almost all)
  (Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, Moriello, Salvatori, Smirnov, 2019)
- 2-loop five-point massless planar and nonplanar integrals
  (Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, 2019)

....

and there would be many more in the near future

uncharted territory

Feynman integral with elliptic polylogarithms,
usually in multi-loop nonplanar diagrams with “large” massive internal loop

new transcendental functions (new, even for mathematicians)

see, eg, Broedel, Duhr, Dulat, Tancredi, 2018

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Main example in this talk

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Mainstream Analytic Methods

Most used

- Canonical differential equation (Henn 2013), in terms of polylagarithm
- Partial fraction + recursive integration (Panzer 2015), package: **HyperInt**, in terms of polylagarithm
- Elliptic Canonical Differetial Equation (Broedel, Duhr, Dulat, Tancredi, 2018), in terms of elliptic polylagarithm

Sometimes magic

- Mellin-Barnes
- Dimension Recursion Relations (Lee, Smirnov 2012),
  usually numeric, but sometimes provide analytic result for complicated integrals
- Integral Bootstrap (Chicherin, Henn, Mitev 2017)
  can easily get the "nice" integrals, eg. conformal, in an integral family

Outline

Latest Developments in Canonical differential equation

- Uniformal Transcendental (UT) integral determination
- Fast differential equation reconstruction
- Solving differential equation with boundary condition

Example: 2-loop 3-jet production analytic integrals and applications in amplitudes

Based on

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia

"*All master integrals for three-jet production at NNLO*", PhysRevLett. 123 (2019), no. 4 041603
"*Analytic result for a two-loop five-particle amplitude*", PhysRevLett. 122 (2019), no. 12 121602
"*The two-loop five-particle amplitude in N= 8 supergravity*", JHEP 1903 (2019) 115

Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, **YZ**, Zoia

"*Analytic form of the full two-loop five-gluon all-plus helicity amplitude*", PhysRevLett. 123 (2019) no.7, 071601

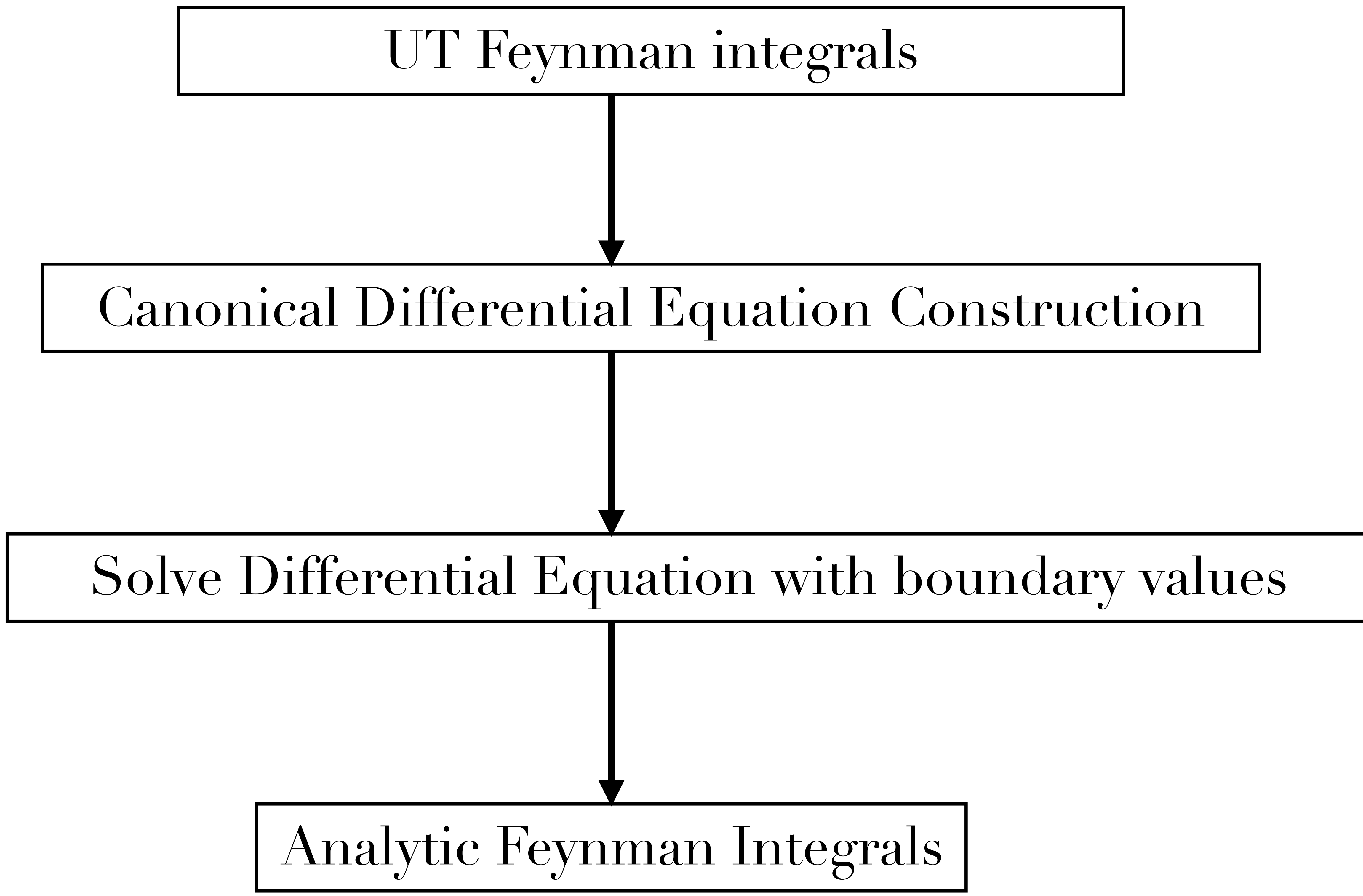
Caron-Huot, Chicherin, Henn, Peraro, **YZ**, Zoia

"*Multi-Regge Limit of the Two-Loop Five-Point Amplitudes in N=4 Super Yang-Mills and N=8 Supergravity* ", 2003.03120 (Accepted in JHEP)

Boehm, Wittmann, Xu, Wu and **YZ**

"*IBP reduction coefficients made simple* " 2008.13194

Canonical Differential Equation for Analytic Feynman integrals



D-dimensional cuts
Module lift
INITIAL algorithm

Alphabet searching
Differential equation reconstruction

Determine boundary values
from the singularity of Feynman integrals

New Ideas

Baikov representation
Computational Algebraic geometry
Finite fields
Large-scale parallelization

Canonical differential equation and UT integrals

Differential Equation for Feynman Integrals

$d = 4 - 2\epsilon$ $I(\bar{x}, \epsilon)$ master integrals as a column vector

$$\frac{\partial}{\partial x_i} I(\bar{x}, \epsilon) = A_i(\bar{x}, \epsilon) I(\bar{x}, \epsilon)$$ (Kotikov 1991)

Can be used both numerically and analytically
eg. Gehrmann, Remiddi 1999, Papadopoulos 2014, Liu, Ma, Wang, 2018 ...

Different choices of the master integrals change the DE dramatically. The simplest choice is the integrals with uniform transcendental (UT) weights, which gives

Canonical Differential Equation

Canonical Differential Equation Henn 2013

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon \tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$
$$= \epsilon \left(\sum_{l=1} \frac{\partial \log(W_l)}{\partial x_i} m_l \right) \tilde{I}(\bar{x},\epsilon)$$

Proportional to $\boldsymbol{\epsilon}$

Symbol letters

Constant rational number matrix

The first line ensures that the equation can be solved perturbatively in $\boldsymbol{\epsilon}$
The second line ensures that the solution is the polylogarithm function in symbol letters

Analogy of the interaction pictures in quantum mechanics

$$\frac{\partial}{\partial x_i} I(\bar{x},\epsilon) = A_i(\bar{x},\epsilon) I(\bar{x},\epsilon)$$

$$\tilde{I}(\bar{x},\epsilon) = T(\bar{x},\epsilon) I(\bar{x},\epsilon)$$

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon \tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = (H_0 + \epsilon H_1)|\psi\rangle$$

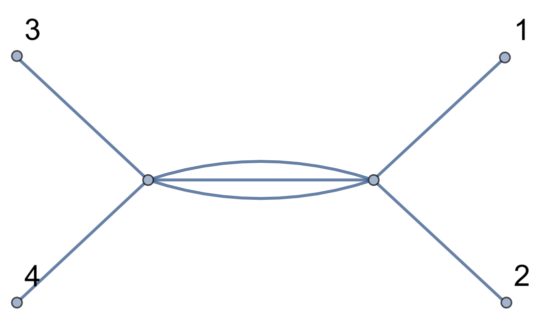
$$|\psi\rangle_I = e^{iH_0 t}|\psi\rangle$$

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle_I = \epsilon H_I(t)|\psi\rangle_I$$

Uniformly transcendental (UT) integrals

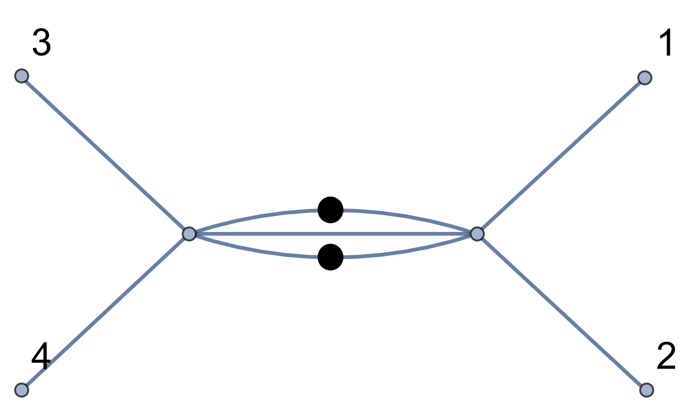
$$\mathcal{T}(\log)=1, \mathcal{T}(\pi)=1, \mathcal{T}(\zeta_n)=n, \mathcal{T}(\mathrm{Li}_n)=n, \ldots, \mathcal{T}(f_1 f_2)=\mathcal{T}(f_1)+\mathcal{T}(f_2)$$

$$I=(\text{overall normalization})\times\sum_{k=0}^{\infty}\epsilon^k f_k, \quad \mathcal{T}(f_k)=k$$



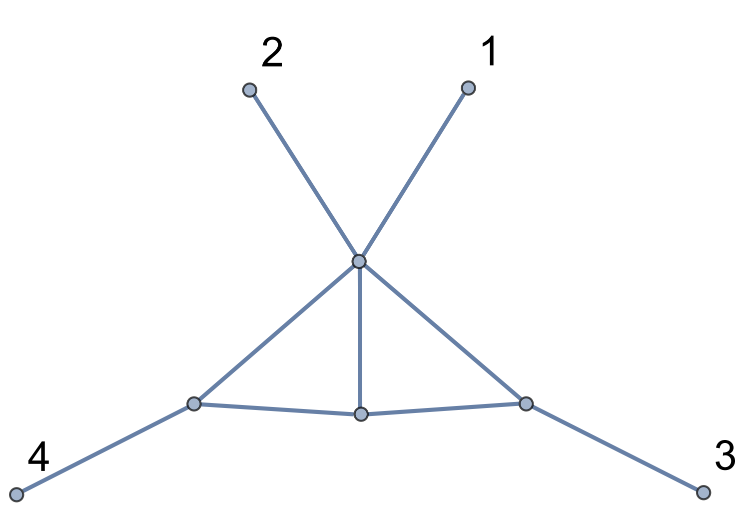
$$(s_{12})^{1-2\epsilon}\left(\frac{1}{4\epsilon}+\frac{13}{8}+\frac{1}{48}\left(345-2\pi^2\right)\epsilon+\frac{1}{96}\left(-256\zeta(3)+2595-26\pi^2\right)\epsilon^2+O\left(\epsilon^3\right)\right)$$

not UT



$$(s_{12})^{-1-2\epsilon}\left(-\frac{1}{\epsilon^2}+\frac{\pi^2}{6}+\frac{32\zeta(3)\epsilon}{3}+\frac{19\pi^4\epsilon^2}{120}+O\left(\epsilon^3\right)\right)$$

UT but not dlog



$$(s_{12})^{-1-2\epsilon}\left(-\frac{1}{4\epsilon^4}+\frac{\pi^2}{24\epsilon^2}+\frac{8\zeta(3)}{3\epsilon}+\frac{19\pi^4}{480}+O\left(\epsilon^1\right)\right)$$

UT and dlog

UT basis is also good for numeric computations

UT Integral ⇒ Canonical DE

$$\tilde{I} = (\text{overall normalization}) \times \sum_{k=0}^{\infty} \epsilon^k f_k, \quad \mathcal{T}(f_k) = k$$

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon \tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$

Feynman integrals are the iterated integration of rational functions ⇒ polylogarithm functions

$$\tilde{I}(x) = P\exp\left(\epsilon \int_{\mathcal{C}} dA\right)\tilde{I}(x_0)$$

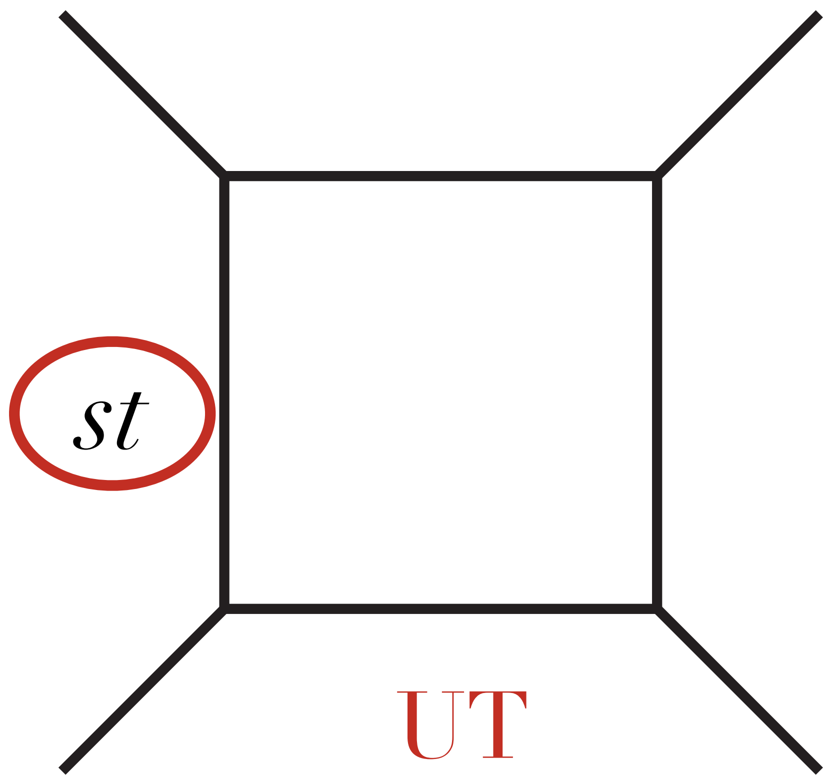
path-ordered

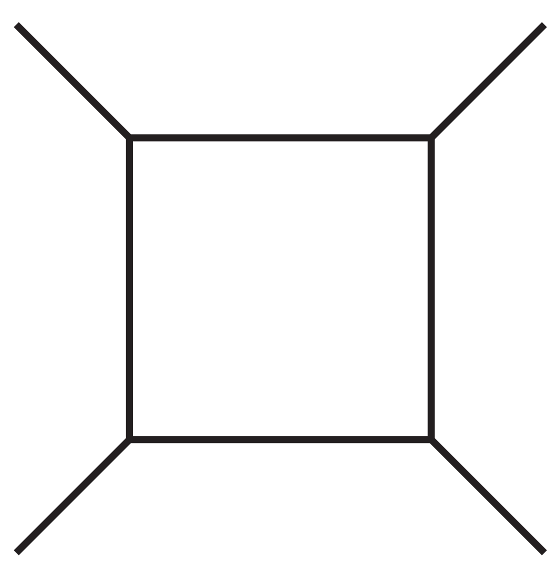
Analogy of perturbation theory of QFT (Dyson series)

Chen's (陈国才) iterated integrals, homotopically invariant

For finite UT integrals, iterated integration is further truncated and simplified. (Caron-Huot, Henn 2014)

To find UT integrals

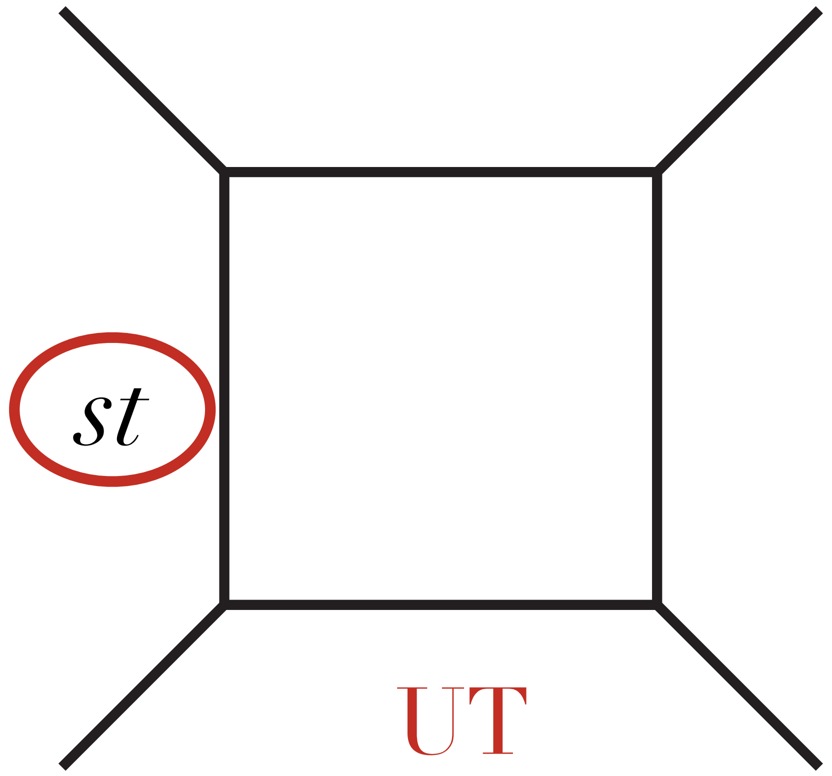




$$= \frac{4}{s\,t\,\epsilon^2} - \frac{2\,\mathrm{Log}\left[\frac{t}{s}\right]}{(s\,t)\,\epsilon} - \frac{4\,\pi^2}{3\,(s\,t)} + \left(\frac{\mathrm{Log}\left[\frac{t}{s}\right]\left(13\,\pi^2 + 6\,\mathrm{Log}\left[\frac{t}{s}\right]\left(i\,\pi + \mathrm{Log}\left[\frac{t}{s}\right]\right)\right)}{6\,s\,t} - \frac{2\,\pi\left(\pi + i\,\mathrm{Log}\left[\frac{t}{s}\right]\right)\mathrm{Log}\left[1+\frac{t}{s}\right]}{s\,t} + \frac{2\,i\,\pi\,\mathrm{Log}\left[1+\frac{t}{s}\right]^2}{s\,t} + \frac{2\,\mathrm{Log}\left[\frac{t}{s}\right]\mathrm{PolyLog}\left[2,-\frac{s}{t}\right]}{s\,t} + \frac{2\,\mathrm{PolyLog}\left[3,1+\frac{s}{t}\right]}{s\,t} + \frac{4\,\mathrm{PolyLog}\left[3,-\frac{t}{s}\right]}{s\,t} + \frac{2\,\mathrm{PolyLog}\left[3,1+\frac{t}{s}\right]}{s\,t} - \frac{40\,\mathrm{Zeta}[3]}{3\,s\,t}\right)\epsilon + O[\epsilon]^2$$

But we want to “guess" UT integrals before we get the analytic result

To find UT integrals: 4D integrand analysis



Integrals may be UT if

- 4D residues (leading singularity) are rational constants
- or 4D integrand can be written as a "dlog" product

Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010

$$\mathrm{Res}\left(\frac{st}{D_1 \ldots D_4}\right) = \pm 1$$

$$st \int d^4 l \frac{1}{D_1 D_2 D_3 D_4} = \int d\log\left(\frac{F}{D_1}\right) \wedge d\log\left(\frac{F}{D_2}\right) \wedge d\log\left(\frac{F}{D_3}\right) \wedge d\log\left(\frac{F}{D_4}\right)$$

Wasser algorithm for dlog (master thesis 2017)

Consider the partial fraction in x_1,

$$\sum_i \frac{dx}{x_1 - a_i} \wedge \Omega_i = \sum_i d\log(x_1 - a_i) \wedge \Omega_i$$

Advanced D-dimensional leading singularity

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, Phys.Rev.Lett. 123 (2019), no. 4 041603

- Consider residue/dlog in D-dimension (Baikov representation)

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = C(s_{ij}, L, E, D) \int dz_1 \ldots dz_k \frac{F^{\frac{D-L-E-1}{2}}}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}}$$ Baikov 1994

- Require that the integral numerators are "almost" polynomials in Mandelstam variables and irreducible scalar product, except for a pseudo scalar as a divisor

$$N = \sum f_\alpha(s_{ij}) \times (\text{scalar product})^\alpha$$

to be solved for

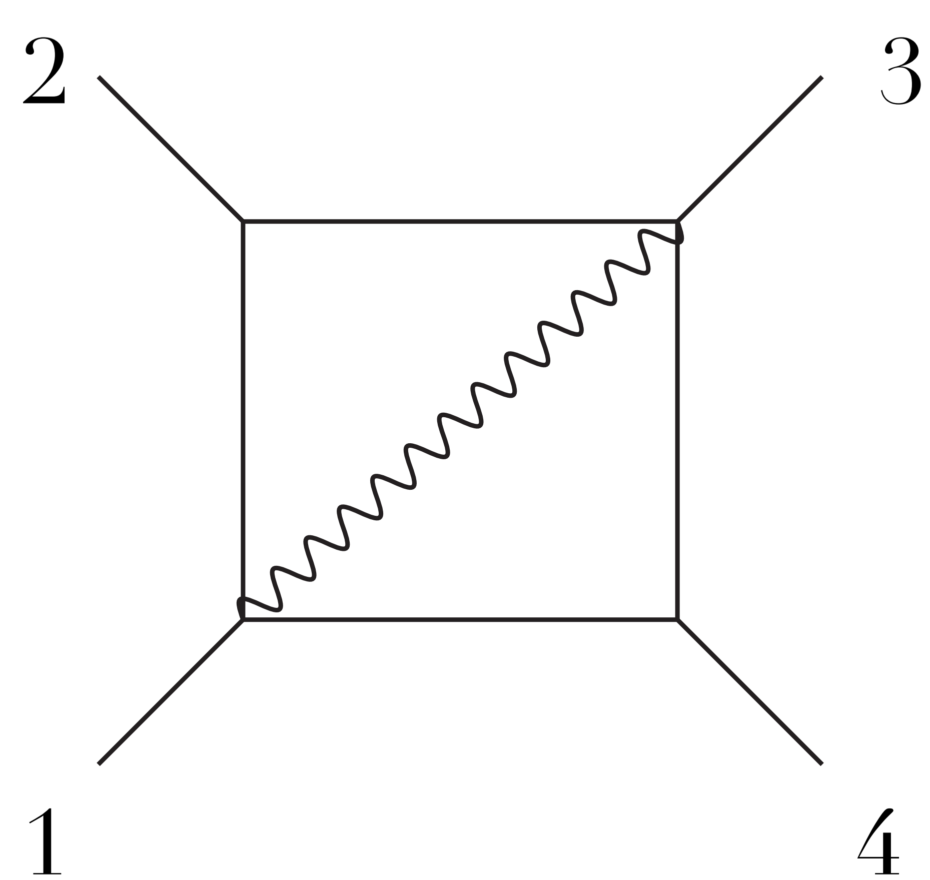
$$\sum_\alpha f_a \times \text{Res}\left(\frac{(\text{Scalar Product})^\alpha}{D_1 \ldots D_n}\right) = (1, 0, \ldots, 0, \ldots 0)$$

Module Lift Method

easily solved by **Singular codes** (computational algebraic geometry software)
systematic analysis and public code on the way!

Other methods to find UT integrals

Chiral Numerator



$$(l - \frac{[34]}{[13]} 4\tilde{1})^2$$

Arkani-Hamed, Bourjaily, Cachazo, Trnka (2012)
Bourjaily, Trnka (2015)

chiral numerator vanishes
in the region the integral becomes IR div.

Integrals with chiral numerators are usually UT

Lee's algorithm (2015)

- First reduce the high-multiplicity poles in DE to get a Fuchsian form
- Use matrix transformation to make the DE proportional to ϵ

Mostly for one-variable cases, heavy computation

Fuchsia (Gituliar, Magerya, 2016), epsilon (Prausa 2016)

New Algorithm: INITIAL

"One UT integral rules all" Dlpha, Henn, Kai 2020

$(f_1, \ldots, f_n)$ is a master integral basis, where f_1 is UT but the others are not.
Try to find a UT basis $(f_1, g_2 \ldots g_n)$ which is UT.

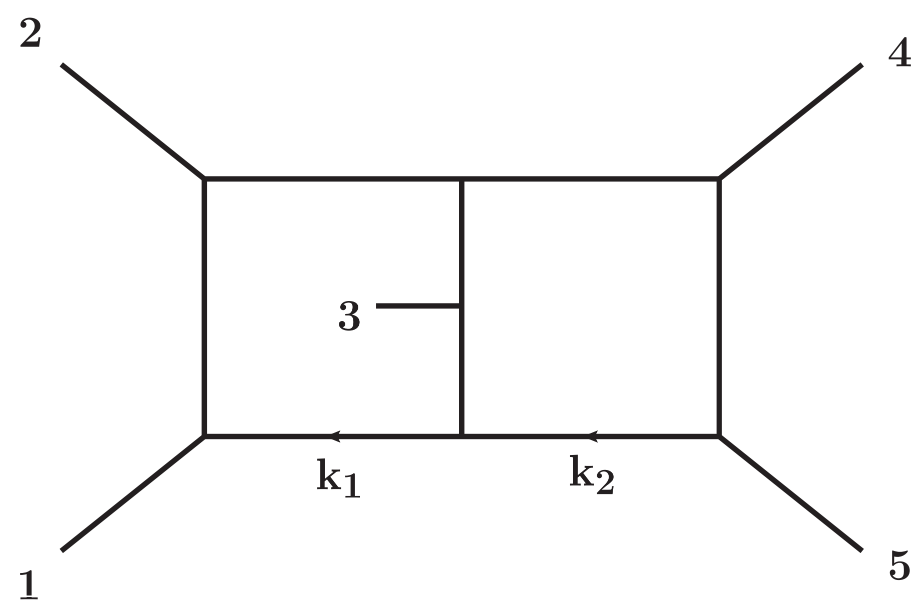
$$\begin{pmatrix} f_1^{[1]} \\ f_1^{[2]} \\ . \\ . \\ f_1^{[n]} \end{pmatrix} = \Psi \begin{pmatrix} f_1 \\ f_2 \\ . \\ . \\ f_n \end{pmatrix} \qquad \begin{pmatrix} f_1^{[1]} \\ f_1^{[2]} \\ . \\ . \\ f_1^{[n]} \end{pmatrix} = \Phi \begin{pmatrix} f_1 \\ g_2 \\ . \\ . \\ g_n \end{pmatrix}$$

$$v_0 \Psi^{-1} \Phi = v_0, \quad v_0 = (1, 0 \ldots 0)$$

make an ansatz with symbol letters, to be solved for

combined with FiniteFlow (Peraro 2019)
a very promising algorithm

Examples: 2-loop 5-point nonplanar UT



Bern, Herrmann, Litsey, Stankowicz, Trnka 2015

$$B[1] = \langle 13\rangle\langle 24\rangle\Bigg([24][13]\Big(-k_2+\frac{[45]}{[24]}\lambda_5\tilde{\lambda_2}\Big)^2\Big(k_1-p_1-\frac{[23]}{[13]}\lambda_2\tilde{\lambda_1}\Big)^2,$$

$$+[14][23]\Big(-k_2+\frac{[45]}{[14]}\lambda_5\tilde{\lambda_1}\Big)^2\Big(k_1-p_2-\frac{[13]}{[23]}\lambda_1\tilde{\lambda_2}\Big)^2\Bigg),$$

$$B[2] = B[1]\Big|_{\substack{p_1\leftrightarrow p_2,\, p_4\leftrightarrow p_5\\ k_1\to -k_1+p_1+p_2,\, k_2\to -k_2-p_4-p_5}},$$

$$B[3] = B[1]\Big|_{\substack{p_1\leftrightarrow p_5,\, p_2\leftrightarrow p_4\\ k_1\to -k_2,\, k_2\to -k_1}},$$

$$B[4] = B[2]\Big|_{\substack{p_1\leftrightarrow p_5,\, p_2\leftrightarrow p_4\\ k_1\to -k_2,\, k_2\to -k_1}},$$

$$B[5] = B[1]^*,\quad B[6] = B[2]^*,\quad B[7] = B[3]^*,\quad B[8] = B[4]^*$$

4D dlog but NOT UT
as verified on differential equations

Examples: 2-loop 5-point nonplanar UT

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, Phys.Rev.Lett. 123 (2019), no. 4 041603

D-dimensional UT integrand analysis: Baikov rep. + Module lift

$$(B[1]+B[5]) + \frac{16 s_{45} G_{12}}{\epsilon_5^2} \times (-s_{12}s_{15} + s_{12}s_{23} + 2s_{12}s_{34} + s_{23}s_{34} + s_{15}s_{45} - s_{34}s_{45})$$

Add terms to make dlog to be UT

$$G_{11} = G\begin{pmatrix} k_1, p_1, p_2, p_3, p_4 \\ k_1, p_1, p_2, p_3, p_4 \end{pmatrix}$$
$$G_{12} = G\begin{pmatrix} k_1, p_1, p_2, p_3, p_4 \\ k_2, p_1, p_2, p_3, p_4 \end{pmatrix}$$
$$G_{22} = G\begin{pmatrix} k_2, p_1, p_2, p_3, p_4 \\ k_2, p_1, p_2, p_3, p_4 \end{pmatrix}.$$

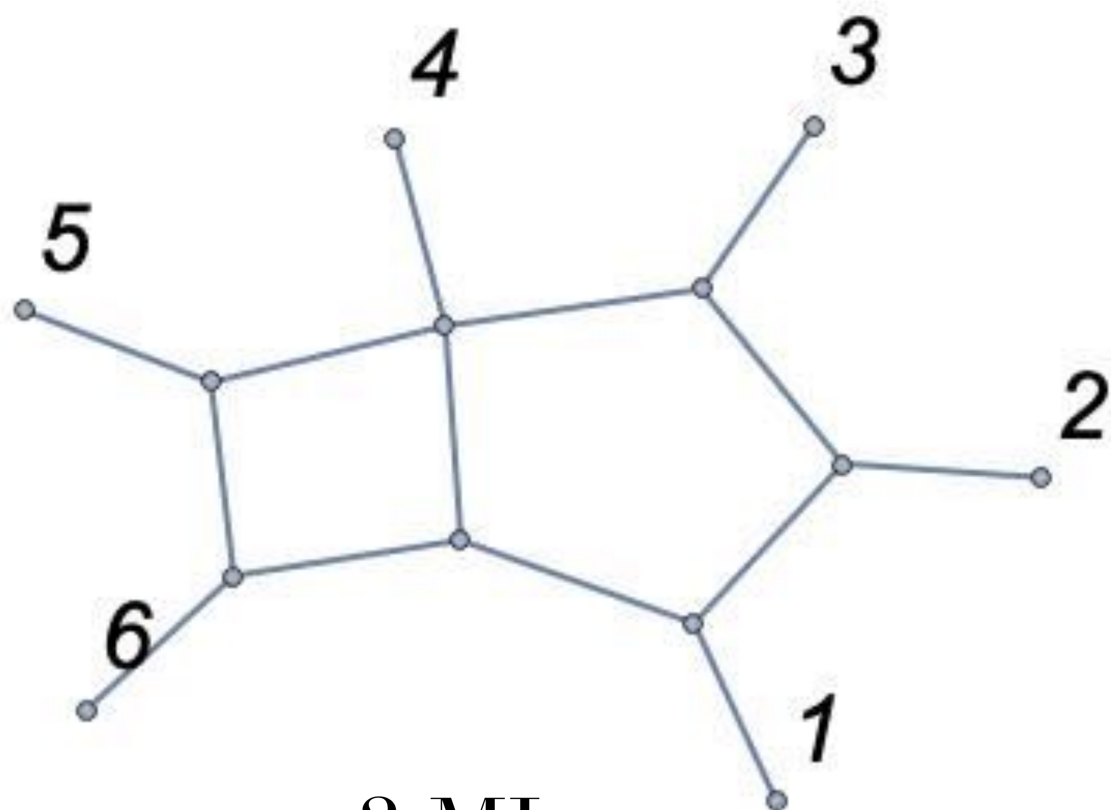
$$\frac{s_{45}}{\epsilon_5}(G_{11} - G_{12}), \quad \frac{s_{12}}{\epsilon_5}(G_{22} - G_{12}), \quad \frac{s_{12} - s_{45}}{\epsilon_5} G_{12}$$

4D vanishing terms, UT

all 2-loop 5-point nonplanar UT integral found in this way

Examples: 2-loop 6-point UT

Henn, Peraro, Xu, **YZ**, to appear



3 MIs

$$N_1^{\text{pentabox}} = s_{12}\left(s_{16} + \frac{\langle 26\rangle[23][61]}{[13]}\right)s_{56}\boxed{(l_1 - w_1)^2}$$

$$N_2^{\text{pentabox}} = s_{12}\left(s_{16} + \frac{[26]\langle 23\rangle\langle 61\rangle}{\langle 13\rangle}\right)s_{56}\boxed{(l_1 - w_2)^2}$$

$$N_3^{\text{pentabox}} = 2s_{12}s_{56}\boxed{l_1 \cdot (w_1 - w_2)}(l_1 + p_6)^2$$

$$w_1 = -\frac{Q_{456}\cdot\tilde\lambda_3\tilde\lambda_1}{[13]}, \quad w_2 = w_1^* .$$

chiral numerators
other factors fixed by D-dimensional integrand analysis

to derive canonical DE
from UT integrals

Symbol letters to full DE

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon\tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$
$$= \epsilon\left(\sum_{l=1}\frac{\partial\log(W_l)}{\partial x_i}m_l\right)\tilde{I}(\bar{x},\epsilon)$$

Symbol letters

Constant rational number matrix

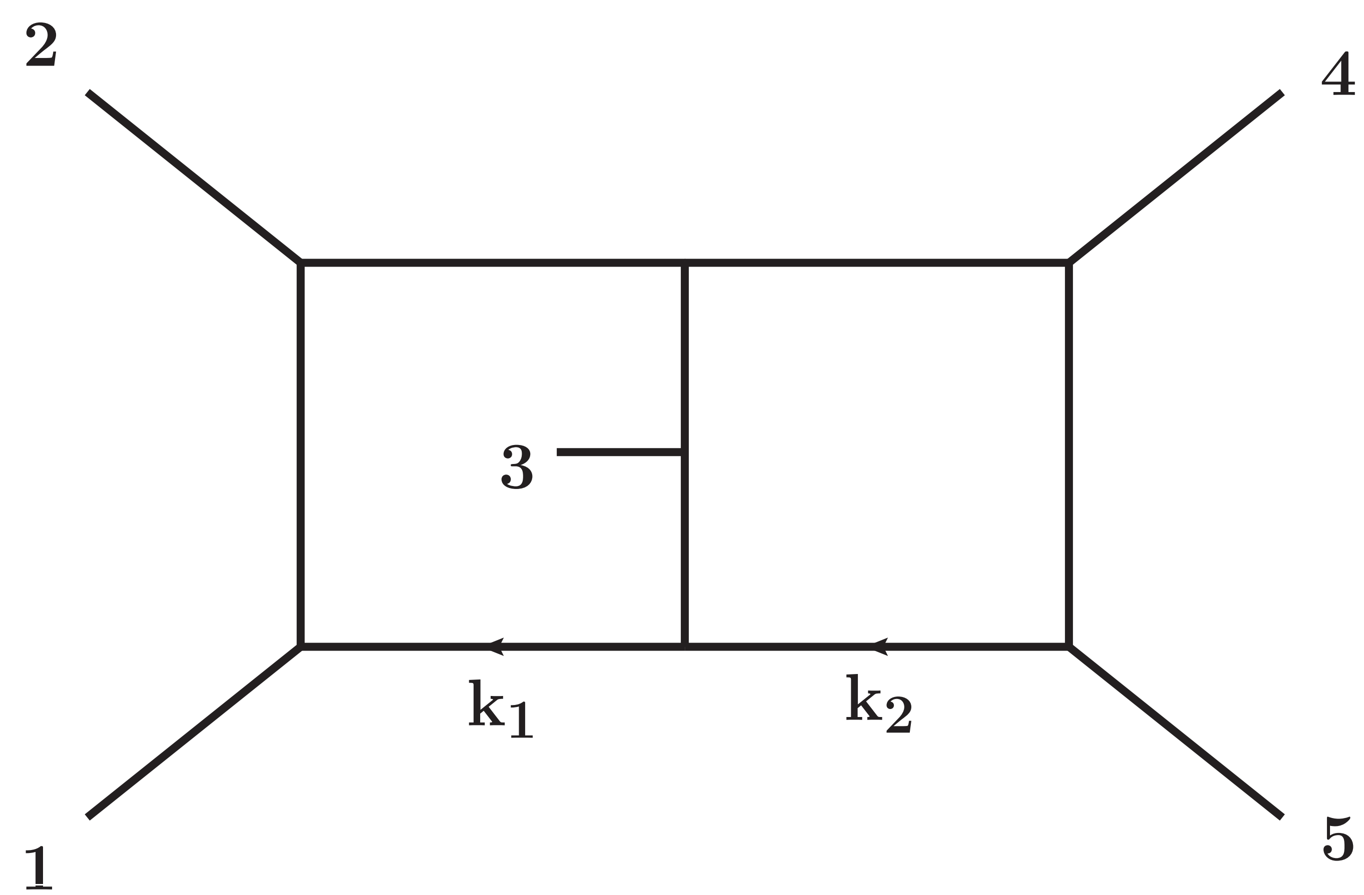
If all the symbol letters W_l are known, then the DE (constant matrix m_l) can be fitted easily by numeric IBPs instead of using analytic IBPs

to find symbol letters:

- Compute DE analytic on the top sector with maximal cut and then permute
- Include the known sub-topology integral symbol letters, lower-loop order symbol letters
- Algorithm to find odd letters in term of even letters (under development)

Examples: 2-loop 5-point nonplanar DE

Traditional way:



$$\frac{\partial}{\partial x_i} I = A_i I$$

1.4 GB

It took 3 months on UZH cluster to do the algebra …

Canonical differential equation and UT:

$$\tilde{I} = T(\epsilon) I, \quad \frac{\partial}{\partial x_i} \tilde{I} = \epsilon \tilde{A}_i \tilde{I}$$

Canonical
5 MB

Examples: 2-loop 5-point nonplanar DE

New method:

$$d\tilde{I}(s_{ij};\epsilon) = \epsilon \left(\sum_{k=1}^{31} a_k d\log W_k(s_{ij}) \right) \tilde{I}(s_{ij};\epsilon)$$

31 (108,108) matrices with rational number entries, fitted numerically, 291 kB

symbol letters

$$\begin{aligned}
&W_1 = v_1\,, && W_6 = v_3+v_4\,, && W_{11} = v_1 - v_4\,, && W_{16} = v_1+v_2-v_4\,,\\
&W_2 = v_2\,, && W_7 = v_4+v_5\,, && W_{12} = v_2 - v_5\,, && W_{17} = v_2+v_3-v_5\,,\\
&W_3 = v_3\,, && W_8 = v_5+v_1\,, && W_{13} = v_3 - v_1\,, && W_{18} = v_3+v_4-v_1\,,\\
&W_4 = v_4\,, && W_9 = v_1+v_2\,, && W_{14} = v_4 - v_2\,, && W_{19} = v_4+v_5-v_2\,,\\
&W_5 = v_5\,, && W_{10} = v_2+v_3\,, && W_{15} = v_5 - v_3\,, && W_{20} = v_5+v_1-v_3\,,
\end{aligned}$$

$$v_1 = s_{12},\ v_2 = s_{23},\ v_3 = s_{34},\ v_4 = s_{45},\ v_5 = s_{15}$$

$$\begin{aligned}
W_{21} &= v_3+v_4-v_1-v_2\,,\\
W_{22} &= v_4+v_5-v_2-v_3\,,\\
W_{23} &= v_5+v_1-v_3-v_4\,,\\
W_{24} &= v_1+v_2-v_4-v_5\,,\\
W_{25} &= v_2+v_3-v_5-v_1\,,
\end{aligned}$$

only for nonplanar

$$\begin{aligned}
W_{26} &= \frac{v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 - \sqrt{\Delta}}{v_1v_2 - v_2v_3 + v_3v_4 - v_1v_5 - v_4v_5 + \sqrt{\Delta}}\,,\\
W_{27} &= \frac{-v_1v_2 + v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 - \sqrt{\Delta}}{-v_1v_2 + v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 + \sqrt{\Delta}}\,,\\
W_{28} &= \frac{-v_1v_2 - v_2v_3 + v_3v_4 + v_1v_5 - v_4v_5 - \sqrt{\Delta}}{-v_1v_2 - v_2v_3 + v_3v_4 + v_1v_5 - v_4v_5 + \sqrt{\Delta}}\,,\\
W_{29} &= \frac{v_1v_2 - v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 - \sqrt{\Delta}}{v_1v_2 - v_2v_3 - v_3v_4 - v_1v_5 + v_4v_5 + \sqrt{\Delta}}\,,\\
W_{30} &= \frac{-v_1v_2 + v_2v_3 - v_3v_4 + v_1v_5 - v_4v_5 - \sqrt{\Delta}}{-v_1v_2 + v_2v_3 - v_3v_4 + v_1v_5 - v_4v_5 + \sqrt{\Delta}}\,,
\end{aligned}$$

odd letters

$$W_{31} = \sqrt{\Delta}\,.$$

FiniteFlow package (Peraro 2019), less than one hour

Solving canonical differential equation

Chen's iterated integrals

$$\tilde{I}(s_{ij}, \epsilon) = \epsilon^{-4} \sum_{m}^{\infty} \epsilon^m \tilde{I}^{(m)}(s_{ij})$$

$$\epsilon^4 \tilde{I}(s_{ij}, \epsilon) = B^{(0)} + \epsilon\left(B^{(1)} + \int_\gamma dA(s_{ij}) B^{(0)}\right) + \epsilon^2\left(B^{(2)} + \int_\gamma dA(s_{ij})\left(B^{(1)} + \int_{\gamma'} dA(s_{ij}) B^{(0)}\right)\right) + \ldots$$

boundary value

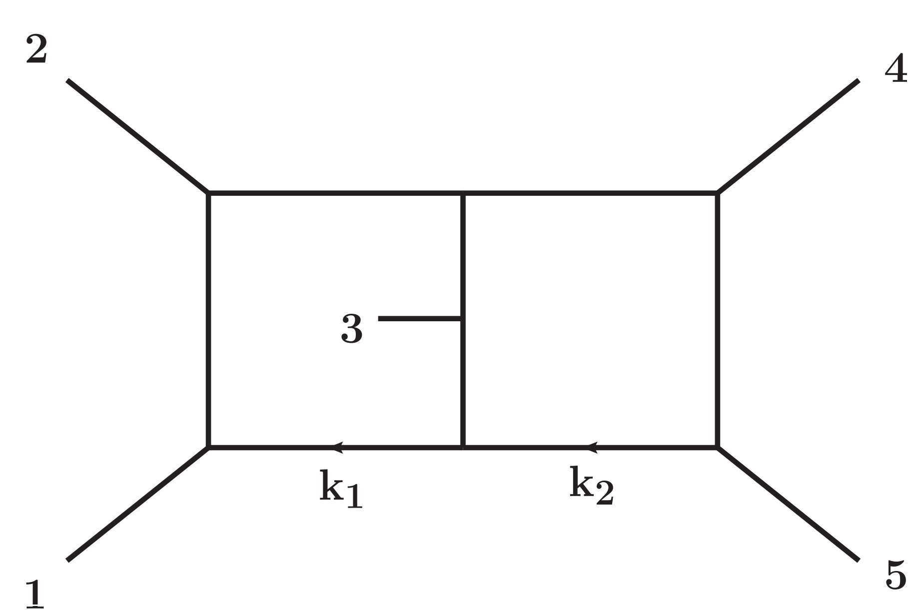
$$\epsilon^4 \tilde{I}(e_{ij}, \epsilon) = \sum_{m=0}^{\infty} \epsilon^m B^{(m)}$$

boundary point

Boundary value

Choice a boundary point

- Must be in the kinematic region under consideration
- Use simple and symmetric numbers



$$\{e_{12}, e_{23}, e_{34}, e_{45}, e_{15}\} = \{3, -1, 1, 1, -1\}$$

108 master integrals → 49 independent integrals

These two conditions usually determine the boundary value analytically

- Many subtopology integrals are known analytically
- "dlog" integral may still be finite when ϵ<0, even if a symbol letter vanishes

Analytic Feynman integral: Solution

implemented in **Ginac**

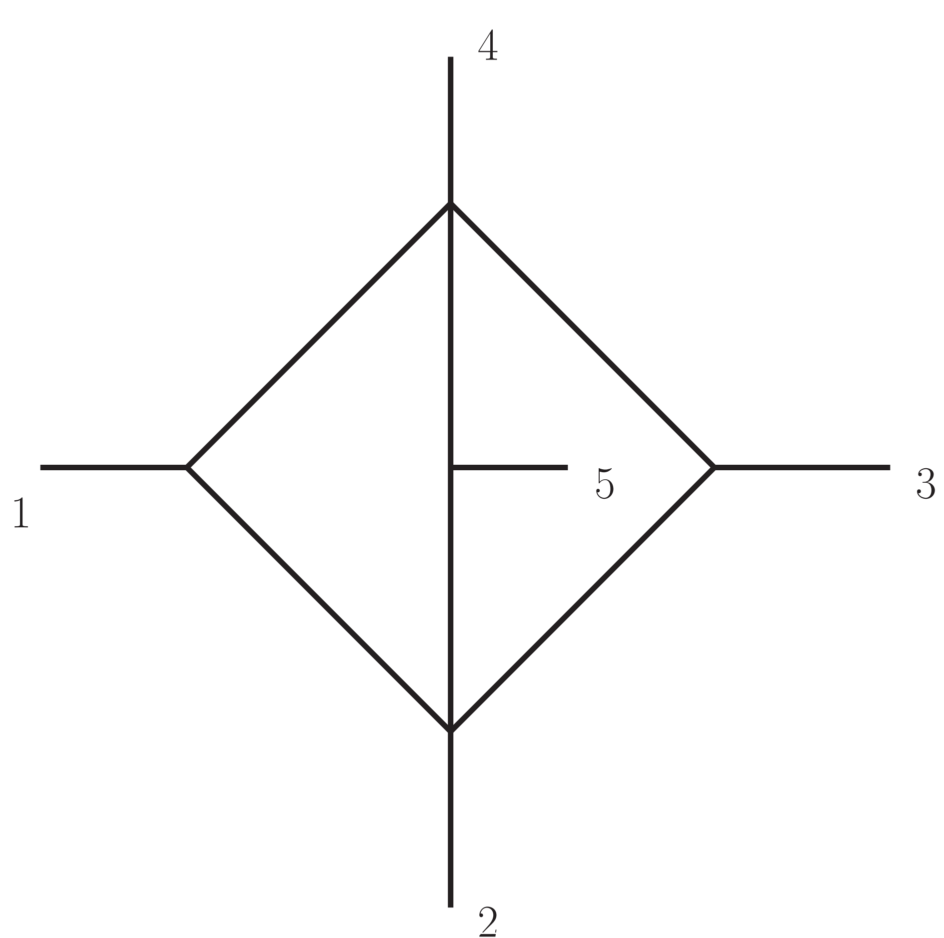
$$G(\underbrace{0,\ldots,0}_{k};z) = \frac{1}{k!}\left(\log z\right)^k, \qquad G(a_1,\ldots,a_k;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_k;t)$$

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, Phys.Rev.Lett. 123 (2019), no. 4 041603

All 2-loop 5-point massless integrals are analytically evaluated
Goncharov polylogarithm (or simpler logarithm) up to weight 4

A tiny example:

Caron-Huot, Chicherin, Henn, Peraro, **YZ**, Zoia 2020.03120



$$I = \frac{1}{\epsilon^2} f^{(2)} + \frac{1}{\epsilon} f^{(3)} + f^{(4)} + \mathcal{O}(\epsilon)\,.$$

Leading part:

$$f^{(2)} = -3\Big[\mathrm{Li}_2\Big(\frac{1}{W_{27}}\Big) - \mathrm{Li}_2\Big(W_{27}\Big) + \mathrm{Li}_2\Big(\frac{1}{W_{28}}\Big) - \mathrm{Li}_2\Big(\frac{1}{W_{27}W_{28}}\Big) - \mathrm{Li}_2\Big(W_{28}\Big) + \mathrm{Li}_2\Big(W_{27}W_{28}\Big)\Big]\,.$$

Again, why analytic

2-loop 5-point massless integrals in the physical region

Analytic (our result)

with one CPU, GiNaC

~ minutes for one point, 10+ digits

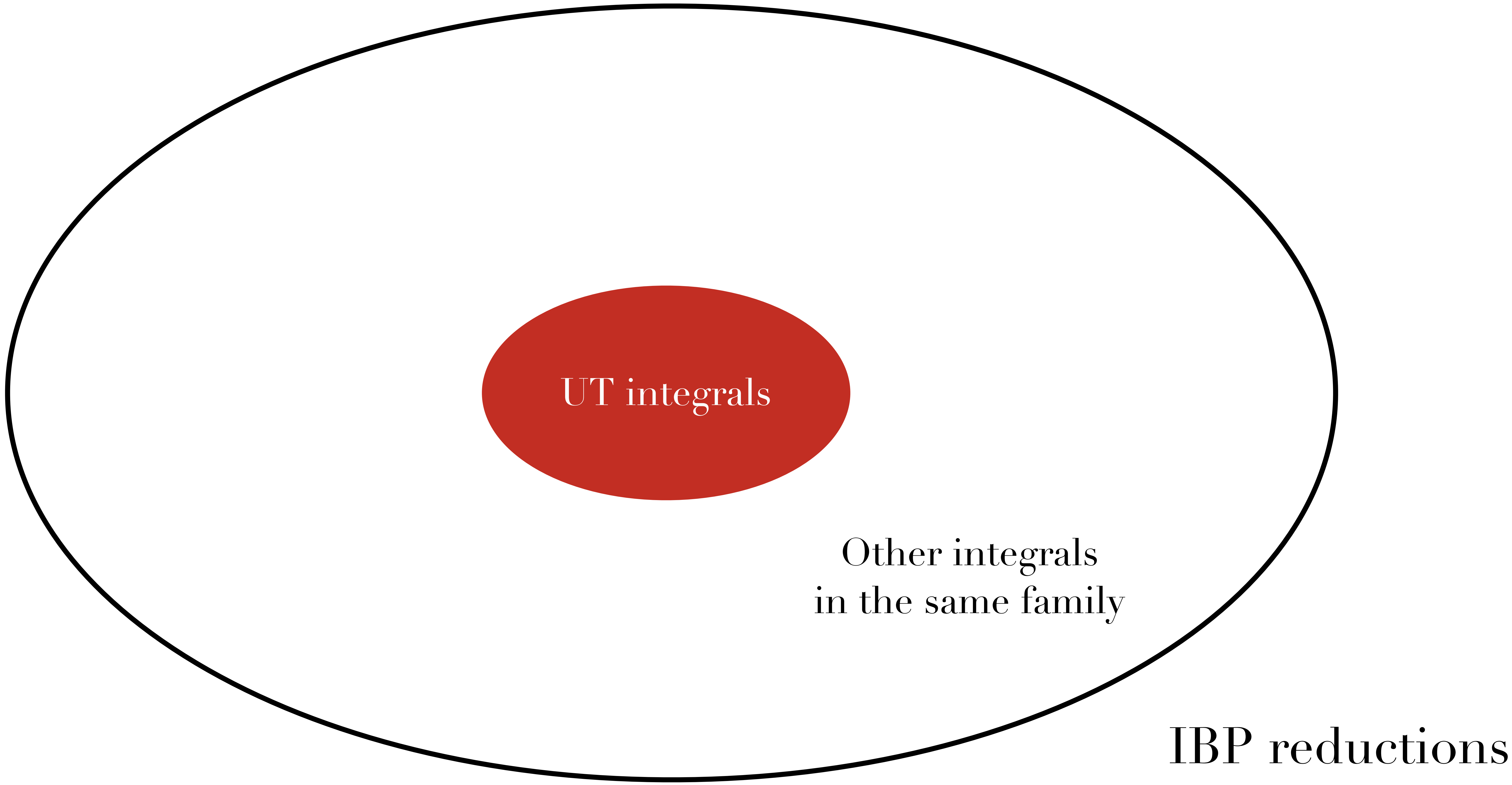
Numeric (**pySecDec**)

$\times 8$

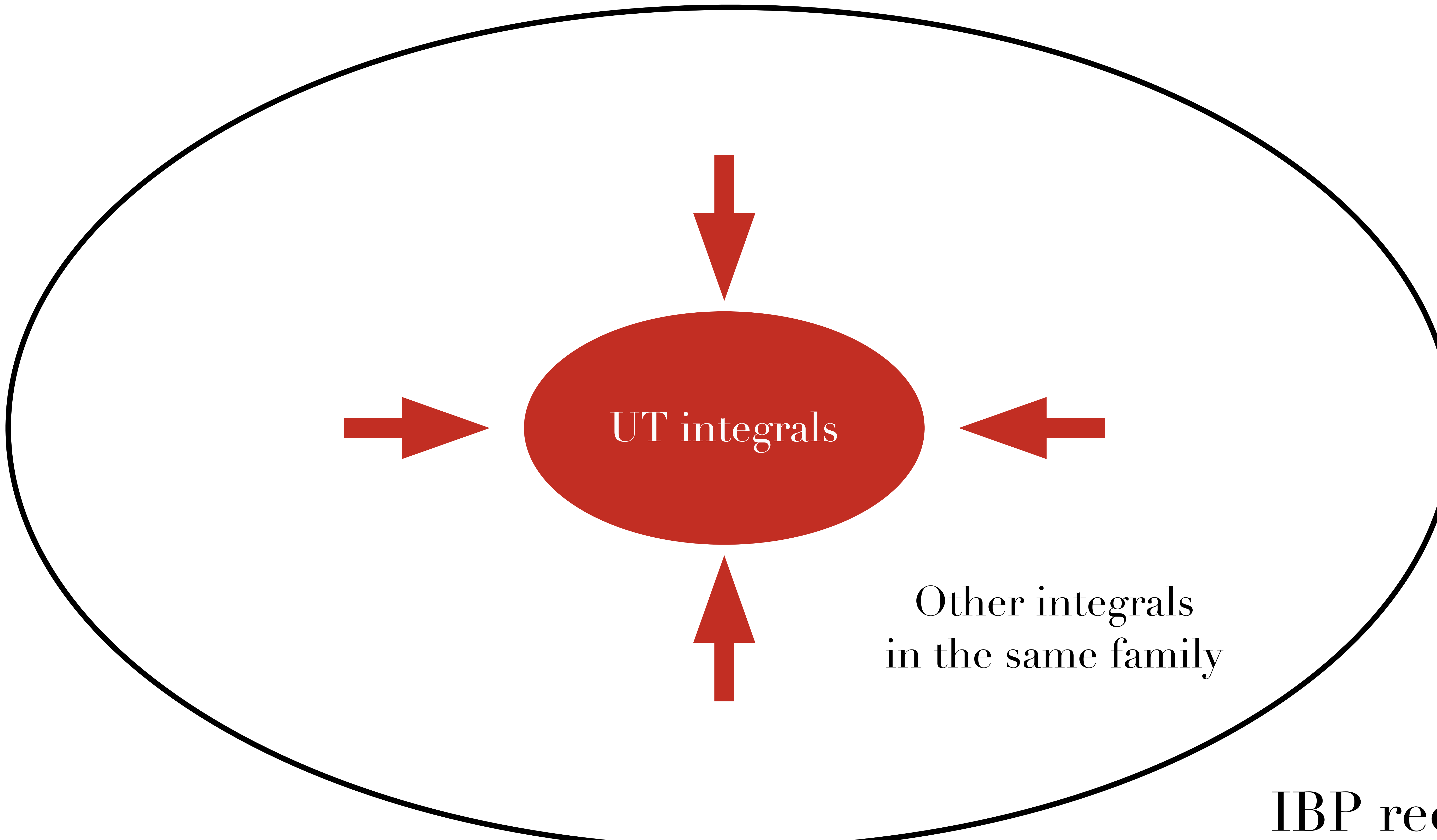
NVIDIA Tesla V100 GPUs

1 week to get one numeric point
error estimated to be ~0.5%

What about the other Feynman integrals?



What about the other Feynman integrals?



IBP reductions

IBP reduction

IBP reduction can reduce **millions** of Feynman integrals to **hundreds** of integrals.

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = 0$$

Chetyrkin, Tkachov 1981

- Laporta algorithm (linear algebra)
- Operator algebra
- Intersection theory (Mastrolia and Mizera, 2018)
- Auxiliary mass expansion (Liu and Ma, 2019)

Typically, IBP reduction coefficients are huge.

IBP reduction coefficients **made simple**

Boehm, Wittmann, Xu, Wu and YZ
2008.13194

In a UT basis, the IBP reduction coefficients have good properties:

- dimensional pole and the kinematic pole separated
- kinematic poles must be (even) letters

Then, we develop a improved Leinartas' algorithm to shorten the coefficients, based on algebraic geometry:

Hilbert's Nullstellensatz, algebraic indepedence, syzygy reduction

We can shorten the coefficients by a factor as large as 10~100.

Look at one example, one 5-page long coefficient

$$
\begin{aligned}
&(-64\epsilon s_{15}^{3}s_{45}^{7}-16s_{15}^{3}s_{45}^{7}+64\epsilon s_{34}^{3}s_{45}^{7}+16s_{34}^{3}s_{45}^{7}-4\epsilon s_{12}s_{15}^{2}s_{45}^{7}-s_{12}s_{15}^{2}s_{45}^{7}\\
&-4\epsilon s_{12}s_{34}^{2}s_{45}^{7}-s_{12}s_{34}^{2}s_{45}^{7}-192\epsilon s_{15}s_{34}^{2}s_{45}^{7}-48s_{15}s_{34}^{2}s_{45}^{7}-48\epsilon s_{23}s_{34}^{2}s_{45}^{7}\\
&-12s_{23}s_{34}^{2}s_{45}^{7}-48\epsilon s_{15}^{2}s_{23}s_{45}^{7}-12s_{15}^{2}s_{23}s_{45}^{7}+12\epsilon s_{12}s_{15}s_{23}s_{45}^{7}+3s_{12}s_{15}s_{23}s_{45}^{7}\\
&+192\epsilon s_{15}^{2}s_{34}s_{45}^{7}+48s_{15}^{2}s_{34}s_{45}^{7}+8\epsilon s_{12}s_{15}s_{34}s_{45}^{7}+2s_{12}s_{15}s_{34}s_{45}^{7}\\
&-12\epsilon s_{12}s_{23}s_{34}s_{45}^{7}-3s_{12}s_{23}s_{34}s_{45}^{7}+96\epsilon s_{15}s_{23}s_{34}s_{45}^{7}+24s_{15}s_{23}s_{34}s_{45}^{7}\\
&-64\epsilon s_{15}^{4}s_{45}^{6}-16s_{15}^{4}s_{45}^{6}+128\epsilon s_{34}^{4}s_{45}^{6}+32s_{34}^{4}s_{45}^{6}+248\epsilon s_{12}s_{15}^{3}s_{45}^{6}\\
&+62s_{12}s_{15}^{3}s_{45}^{6}-196\epsilon s_{12}s_{34}^{3}s_{45}^{6}-49s_{12}s_{34}^{3}s_{45}^{6}-320\epsilon s_{15}s_{34}^{3}s_{45}^{6}\\
&-80s_{15}s_{34}^{3}s_{45}^{6}-352\epsilon s_{23}s_{34}^{3}s_{45}^{6}-76s_{23}s_{34}^{3}s_{45}^{6}+12\epsilon s_{12}^{2}s_{15}^{2}s_{45}^{6}\\
&+3s_{12}^{2}s_{15}^{2}s_{45}^{6}-12\epsilon s_{12}^{2}s_{23}^{2}s_{45}^{6}-3s_{12}^{2}s_{23}^{2}s_{45}^{6}+96\epsilon s_{15}^{2}s_{23}^{2}s_{45}^{6}\\
&+24s_{15}^{2}s_{23}^{2}s_{45}^{6}-96\epsilon s_{12}s_{15}s_{23}^{2}s_{45}^{6}-24s_{12}s_{15}s_{23}^{2}s_{45}^{6}+8\epsilon s_{12}^{2}s_{34}^{2}s_{45}^{6}\\
&+2s_{12}^{2}s_{34}^{2}s_{45}^{6}+192\epsilon s_{15}^{2}s_{34}^{2}s_{45}^{6}+48s_{15}^{2}s_{34}^{2}s_{45}^{6}+144\epsilon s_{23}^{2}s_{34}^{2}s_{45}^{6}\\
&+36s_{23}^{2}s_{34}^{2}s_{45}^{6}+640\epsilon s_{12}s_{15}s_{34}^{2}s_{45}^{6}+160s_{12}s_{15}s_{34}^{2}s_{45}^{6}+264\epsilon s_{12}s_{23}s_{34}^{2}s_{45}^{6}\\
&+54s_{12}s_{23}s_{34}^{2}s_{45}^{6}+784\epsilon s_{15}s_{23}s_{34}^{2}s_{45}^{6}+172s_{15}s_{23}s_{34}^{2}s_{45}^{6}+80\epsilon s_{15}^{3}s_{23}s_{45}^{6}\\
&+20s_{15}^{3}s_{23}s_{45}^{6}+176\epsilon s_{12}s_{15}^{2}s_{23}s_{45}^{6}+44s_{12}s_{15}^{2}s_{23}s_{45}^{6}-24\epsilon s_{12}^{2}s_{15}s_{23}s_{45}^{6}\\
&-6s_{12}^{2}s_{15}s_{23}s_{45}^{6}+64\epsilon s_{15}^{3}s_{34}s_{45}^{6}+16s_{15}^{3}s_{34}s_{45}^{6}-692\epsilon s_{12}s_{15}^{2}s_{34}s_{45}^{6}\\
&-173s_{12}s_{15}^{2}s_{34}s_{45}^{6}-12\epsilon s_{12}s_{23}^{2}s_{34}s_{45}^{6}-3s_{12}s_{23}^{2}s_{34}s_{45}^{6}-240\epsilon s_{15}s_{23}^{2}s_{34}s_{45}^{6}\\
&-60s_{15}s_{23}^{2}s_{34}s_{45}^{6}-20\epsilon s_{12}^{2}s_{15}s_{34}s_{45}^{6}-5s_{12}^{2}s_{15}s_{34}s_{45}^{6}+28\epsilon s_{12}^{2}s_{23}s_{34}s_{45}^{6}\\
&+7s_{12}^{2}s_{23}s_{34}s_{45}^{6}-512\epsilon s_{15}^{2}s_{23}s_{34}s_{45}^{6}-116s_{15}^{2}s_{23}s_{34}s_{45}^{6}-440\epsilon s_{12}s_{15}s_{23}s_{34}s_{45}^{6}\\
&-98s_{12}s_{15}s_{23}s_{34}s_{45}^{6}+64\epsilon s_{34}^{5}s_{45}^{5}+16s_{34}^{5}s_{45}^{5}+252\epsilon s_{12}s_{15}^{4}s_{45}^{5}+63s_{12}s_{15}^{4}s_{45}^{5}\\
&-376\epsilon s_{12}s_{34}^{4}s_{45}^{5}-94s_{12}s_{34}^{4}s_{45}^{5}-64\epsilon s_{15}s_{34}^{4}s_{45}^{5}-16s_{15}s_{34}^{4}s_{45}^{5}-560\epsilon s_{23}s_{34}^{4}s_{45}^{5}\\
&-116s_{23}s_{34}^{4}s_{45}^{5}-360\epsilon s_{12}^{2}s_{15}^{3}s_{45}^{5}-90s_{12}^{2}s_{15}^{3}s_{45}^{5}+24\epsilon s_{12}^{2}s_{23}^{3}s_{45}^{5}\\
&+6s_{12}^{2}s_{23}^{3}s_{45}^{5}-48\epsilon s_{15}^{2}s_{23}^{3}s_{45}^{5}-12s_{15}^{2}s_{23}^{3}s_{45}^{5}+156\epsilon s_{12}s_{15}s_{23}^{3}s_{45}^{5}\\
&+39s_{12}s_{15}s_{23}^{3}s_{45}^{5}+256\epsilon s_{12}^{2}s_{34}^{3}s_{45}^{5}+64s_{12}^{2}s_{34}^{3}s_{45}^{5}-192\epsilon s_{15}^{2}s_{34}^{3}s_{45}^{5}\\
&-48s_{15}^{2}s_{34}^{3}s_{45}^{5}+672\epsilon s_{23}^{2}s_{34}^{3}s_{45}^{5}+144s_{23}^{2}s_{34}^{3}s_{45}^{5}+1056\epsilon s_{12}s_{15}s_{34}^{3}s_{45}^{5}\\
&+264s_{12}s_{15}s_{34}^{3}s_{45}^{5}+1164\epsilon s_{12}s_{23}s_{34}^{3}s_{45}^{5}+231s_{12}s_{23}s_{34}^{3}s_{45}^{5}+1088\epsilon s_{15}s_{23}s_{34}^{3}s_{45}^{5}\\
&+236s_{15}s_{23}s_{34}^{3}s_{45}^{5}-12\epsilon s_{12}^{3}s_{15}^{2}s_{45}^{5}-3s_{12}^{3}s_{15}^{2}s_{45}^{5}+32\epsilon s_{12}^{3}s_{23}^{2}s_{45}^{5}\\
&+8s_{12}^{3}s_{23}^{2}s_{45}^{5}-32\epsilon s_{15}^{3}s_{23}^{2}s_{45}^{5}-8s_{15}^{3}s_{23}^{2}s_{45}^{5}-368\epsilon s_{12}s_{15}^{2}s_{23}^{2}s_{45}^{5}\\
&-92s_{12}s_{15}^{2}s_{23}^{2}s_{45}^{5}+236\epsilon s_{12}^{2}s_{15}s_{23}^{2}s_{45}^{5}+59s_{12}^{2}s_{15}s_{23}^{2}s_{45}^{5}-4\epsilon s_{12}^{3}s_{34}^{2}s_{45}^{5}\\
&-s_{12}^{3}s_{34}^{2}s_{45}^{5}+320\epsilon s_{15}^{3}s_{34}^{2}s_{45}^{5}+80s_{15}^{3}s_{34}^{2}s_{45}^{5}-144\epsilon s_{23}^{3}s_{34}^{2}s_{45}^{5}\\
&-36s_{23}^{3}s_{34}^{2}s_{45}^{5}-732\epsilon s_{12}s_{15}^{2}s_{34}^{2}s_{45}^{5}-183s_{12}s_{15}^{2}s_{34}^{2}s_{45}^{5}-832\epsilon s_{12}s_{23}^{2}s_{34}^{2}s_{45}^{5}\\
&-184s_{12}s_{23}^{2}s_{34}^{2}s_{45}^{5}-1152\epsilon s_{15}s_{23}^{2}s_{34}^{2}s_{45}^{5}-240s_{15}s_{23}^{2}s_{34}^{2}s_{45}^{5}\\
&-884\epsilon s_{12}^{2}s_{15}s_{34}^{2}s_{45}^{5}-221s_{12}^{2}s_{15}s_{34}^{2}s_{45}^{5}-464\epsilon s_{12}^{2}s_{23}s_{34}^{2}s_{45}^{5}\\
&-80s_{12}^{2}s_{23}s_{34}^{2}s_{45}^{5}-416\epsilon s_{15}^{2}s_{23}s_{34}^{2}s_{45}^{5}-104s_{15}^{2}s_{23}s_{34}^{2}s_{45}^{5}\\
&-2208\epsilon s_{12}s_{15}s_{23}s_{34}^{2}s_{45}^{5}-468s_{12}s_{15}s_{23}s_{34}^{2}s_{45}^{5}+80\epsilon s_{15}^{4}s_{23}s_{45}^{5}\\
&+20s_{15}^{4}s_{23}s_{45}^{5}-264\epsilon s_{12}s_{15}^{3}s_{23}s_{45}^{5}-66s_{12}s_{15}^{3}s_{23}s_{45}^{5}-220\epsilon s_{12}^{2}s_{15}^{2}s_{23}s_{45}^{5}\\
&-55s_{12}^{2}s_{15}^{2}s_{23}s_{45}^{5}+4\epsilon s_{12}^{3}s_{15}s_{23}s_{45}^{5}+s_{12}^{3}s_{15}s_{23}s_{45}^{5}-128\epsilon s_{15}^{4}s_{34}s_{45}^{5}\\
&-32s_{15}^{4}s_{34}s_{45}^{5}-200\epsilon s_{12}s_{15}^{3}s_{34}s_{45}^{5}-50s_{12}s_{15}^{3}s_{34}s_{45}^{5}+60\epsilon s_{12}s_{23}^{3}s_{34}s_{45}^{5}\\
&+15s_{12}s_{23}^{3}s_{34}s_{45}^{5}+192\epsilon s_{15}s_{23}^{3}s_{34}s_{45}^{5}+48s_{15}s_{23}^{3}s_{34}s_{45}^{5}+988\epsilon s_{12}^{2}s_{15}^{2}s_{34}s_{45}^{5}\\
&+247s_{12}^{2}s_{15}^{2}s_{34}s_{45}^{5}+128\epsilon s_{12}^{2}s_{23}^{2}s_{34}s_{45}^{5}+32s_{12}^{2}s_{23}^{2}s_{34}s_{45}^{5}\\
&+512\epsilon s_{15}^{2}s_{23}^{2}s_{34}s_{45}^{5}+104s_{15}^{2}s_{23}^{2}s_{34}s_{45}^{5}+796\epsilon s_{12}s_{15}s_{23}^{2}s_{34}s_{45}^{5}\\
&+175s_{12}s_{15}s_{23}^{2}s_{34}s_{45}^{5}+16\epsilon s_{12}^{3}s_{15}s_{34}s_{45}^{5}+4s_{12}^{3}s_{15}s_{34}s_{45}^{5}-20\epsilon s_{12}^{3}s_{23}s_{34}s_{45}^{5}\\
&-5s_{12}^{3}s_{23}s_{34}s_{45}^{5}-192\epsilon s_{15}^{3}s_{23}s_{34}s_{45}^{5}-36s_{15}^{3}s_{23}s_{34}s_{45}^{5}+1308\epsilon s_{12}s_{15}^{2}s_{23}s_{34}s_{45}^{5}\\
&+303s_{12}s_{15}^{2}s_{23}s_{34}s_{45}^{5}+632\epsilon s_{12}^{2}s_{15}s_{23}s_{34}s_{45}^{5}+122s_{12}^{2}s_{15}s_{23}s_{34}s_{45}^{5}\\
&-184\epsilon s_{12}s_{34}^{5}s_{45}^{4}-46s_{12}s_{34}^{5}s_{45}^{4}+64\epsilon s_{15}s_{34}^{5}s_{45}^{4}+16s_{15}s_{34}^{5}s_{45}^{4}-256\epsilon s_{23}s_{34}^{5}s_{45}^{4}\\
&-52s_{23}s_{34}^{5}s_{45}^{4}-372\epsilon s_{12}^{2}s_{15}^{4}s_{45}^{4}-93s_{12}^{2}s_{15}^{4}s_{45}^{4}-12\epsilon s_{12}^{2}s_{23}^{4}s_{45}^{4}\\
&-3s_{12}^{2}s_{23}^{4}s_{45}^{4}-72\epsilon s_{12}s_{15}s_{23}^{4}s_{45}^{4}-18s_{12}s_{15}s_{23}^{4}s_{45}^{4}+368\epsilon s_{12}^{2}s_{34}^{4}s_{45}^{4}
\end{aligned}
$$

page 1

Look at one example, one 5-page long coefficient

$$+92s_{12}^2s_{34}^4s_{45}^4-192\epsilon s_{15}^2s_{34}^4s_{45}^4-48s_{15}^2s_{34}^4s_{45}^4+912\epsilon s_{23}^2s_{34}^4s_{45}^4$$
$$+180s_{23}^2s_{34}^4s_{45}^4+240\epsilon s_{12}s_{15}s_{34}^4s_{45}^4+60s_{12}s_{15}s_{34}^4s_{45}^4+1520\epsilon s_{12}s_{23}s_{34}^4s_{45}^4$$
$$+308s_{12}s_{23}s_{34}^4s_{45}^4+208\epsilon s_{15}s_{23}s_{34}^4s_{45}^4+52s_{15}s_{23}s_{34}^4s_{45}^4+232\epsilon s_{12}^3s_{15}^3s_{45}^4$$
$$+58s_{12}^3s_{15}^3s_{45}^4-52\epsilon s_{12}^3s_{23}^3s_{45}^4-13s_{12}^3s_{23}^3s_{45}^4+148\epsilon s_{12}s_{15}^2s_{23}^3s_{45}^4$$
$$+37s_{12}s_{15}^2s_{23}^3s_{45}^4-384\epsilon s_{12}^2s_{15}s_{23}^3s_{45}^4-96s_{12}^2s_{15}s_{23}^3s_{45}^4-124\epsilon s_{12}^3s_{34}^3s_{45}^4$$
$$-31s_{12}^3s_{34}^3s_{45}^4+192\epsilon s_{15}^3s_{34}^3s_{45}^4+48s_{15}^3s_{34}^3s_{45}^4-544\epsilon s_{23}^3s_{34}^3s_{45}^4$$
$$-124s_{23}^3s_{34}^3s_{45}^4+572\epsilon s_{12}s_{15}^2s_{34}^3s_{45}^4+143s_{12}s_{15}^2s_{34}^3s_{45}^4-2284\epsilon s_{12}s_{23}^2s_{34}^3s_{45}^4$$
$$-463s_{12}s_{23}^2s_{34}^3s_{45}^4-1392\epsilon s_{15}s_{23}^2s_{34}^3s_{45}^4-264s_{15}s_{23}^2s_{34}^3s_{45}^4$$
$$-1160\epsilon s_{12}^2s_{15}s_{34}^3s_{45}^4-290s_{12}^2s_{15}s_{34}^3s_{45}^4-1668\epsilon s_{12}^2s_{23}s_{34}^3s_{45}^4$$
$$-321s_{12}^2s_{23}s_{34}^3s_{45}^4+512\epsilon s_{15}^2s_{23}s_{34}^3s_{45}^4+92s_{15}^2s_{23}s_{34}^3s_{45}^4$$
$$-2708\epsilon s_{12}s_{15}s_{23}s_{34}^3s_{45}^4-605s_{12}s_{15}s_{23}s_{34}^3s_{45}^4+4\epsilon s_{12}^4s_{15}^2s_{45}^4+s_{12}^4s_{15}^2s_{45}^4$$
$$-28\epsilon s_{12}^4s_{23}^2s_{45}^4-7s_{12}^4s_{23}^2s_{45}^4+176\epsilon s_{12}s_{15}^3s_{23}^2s_{45}^4+44s_{12}s_{15}^3s_{23}^2s_{45}^4$$
$$+588\epsilon s_{12}^2s_{15}^2s_{23}^2s_{45}^4+147s_{12}^2s_{15}^2s_{23}^2s_{45}^4-156\epsilon s_{12}^3s_{15}s_{23}^2s_{45}^4$$
$$-39s_{12}^3s_{15}s_{23}^2s_{45}^4-64\epsilon s_{15}^4s_{34}^2s_{45}^4-16s_{15}^4s_{34}^2s_{45}^4+48\epsilon s_{23}^4s_{34}^2s_{45}^4$$
$$+12s_{23}^4s_{34}^2s_{45}^4-1128\epsilon s_{12}s_{15}^3s_{34}^2s_{45}^4-282s_{12}s_{15}^3s_{34}^2s_{45}^4+920\epsilon s_{12}s_{23}^3s_{34}^2s_{45}^4$$
$$+218s_{12}s_{23}^3s_{34}^2s_{45}^4+720\epsilon s_{15}s_{23}^3s_{34}^2s_{45}^4+156s_{15}s_{23}^3s_{34}^2s_{45}^4$$
$$+832\epsilon s_{12}^2s_{15}^2s_{34}^2s_{45}^4+208s_{12}^2s_{15}^2s_{34}^2s_{45}^4+1508\epsilon s_{12}^2s_{23}^2s_{34}^2s_{45}^4$$
$$+317s_{12}^2s_{23}^2s_{34}^2s_{45}^4+416\epsilon s_{15}^2s_{23}^2s_{34}^2s_{45}^4+80s_{15}^2s_{23}^2s_{34}^2s_{45}^4$$
$$+2628\epsilon s_{12}s_{15}s_{23}^2s_{34}^2s_{45}^4+501s_{12}s_{15}s_{23}^2s_{34}^2s_{45}^4+560\epsilon s_{12}^3s_{15}s_{34}^2s_{45}^4$$
$$+140s_{12}^3s_{15}s_{34}^2s_{45}^4+460\epsilon s_{12}^3s_{23}s_{34}^2s_{45}^4+79s_{12}^3s_{23}s_{34}^2s_{45}^4$$
$$-624\epsilon s_{15}^3s_{23}s_{34}^2s_{45}^4-132s_{15}^3s_{23}s_{34}^2s_{45}^4+1124\epsilon s_{12}s_{15}^2s_{23}s_{34}^2s_{45}^4$$
$$+317s_{12}s_{15}^2s_{23}s_{34}^2s_{45}^4+2424\epsilon s_{12}^2s_{15}s_{23}s_{34}^2s_{45}^4+510s_{12}^2s_{15}s_{23}s_{34}^2s_{45}^4$$
$$-252\epsilon s_{12}s_{15}^4s_{23}s_{45}^4-63s_{12}s_{15}^4s_{23}s_{45}^4+292\epsilon s_{12}^2s_{15}^3s_{23}s_{45}^4$$
$$+73s_{12}^2s_{15}^3s_{23}s_{45}^4+80\epsilon s_{12}^3s_{15}^2s_{23}s_{45}^4+20s_{12}^3s_{15}^2s_{23}s_{45}^4+16\epsilon s_{12}^4s_{15}s_{23}s_{45}^4$$
$$+4s_{12}^4s_{15}s_{23}s_{45}^4+500\epsilon s_{12}s_{15}^4s_{34}s_{45}^4+125s_{12}s_{15}^4s_{34}s_{45}^4-36\epsilon s_{12}s_{23}^4s_{34}s_{45}^4$$
$$-9s_{12}s_{23}^4s_{34}s_{45}^4-48\epsilon s_{15}s_{23}^4s_{34}s_{45}^4-12s_{15}s_{23}^4s_{34}s_{45}^4+332\epsilon s_{12}^2s_{15}^3s_{34}s_{45}^4$$
$$+83s_{12}^2s_{15}^3s_{34}s_{45}^4-324\epsilon s_{12}^2s_{23}^3s_{34}s_{45}^4-81s_{12}^2s_{23}^3s_{34}s_{45}^4$$
$$-176\epsilon s_{15}^2s_{23}^3s_{34}s_{45}^4-32s_{15}^2s_{23}^3s_{34}s_{45}^4-400\epsilon s_{12}s_{15}s_{23}^3s_{34}s_{45}^4$$
$$-88s_{12}s_{15}s_{23}^3s_{34}s_{45}^4-668\epsilon s_{12}^3s_{15}^2s_{34}s_{45}^4-167s_{12}^3s_{15}^2s_{34}s_{45}^4$$
$$-228\epsilon s_{12}^3s_{23}^2s_{34}s_{45}^4-57s_{12}^3s_{23}^2s_{34}s_{45}^4+64\epsilon s_{15}^3s_{23}^2s_{34}s_{45}^4+4s_{15}^3s_{23}^2s_{34}s_{45}^4$$
$$-1068\epsilon s_{12}s_{15}^2s_{23}^2s_{34}s_{45}^4-219s_{12}s_{15}^2s_{23}^2s_{34}s_{45}^4-664\epsilon s_{12}^2s_{15}s_{23}^2s_{34}s_{45}^4$$
$$-106s_{12}^2s_{15}s_{23}^2s_{34}s_{45}^4-4\epsilon s_{12}^4s_{15}s_{34}s_{45}^4-s_{12}^4s_{15}s_{34}s_{45}^4+4\epsilon s_{12}^4s_{23}s_{34}s_{45}^4$$
$$+s_{12}^4s_{23}s_{34}s_{45}^4+160\epsilon s_{15}^4s_{23}s_{34}s_{45}^4+40s_{15}^4s_{23}s_{34}s_{45}^4+316\epsilon s_{12}s_{15}^3s_{23}s_{34}s_{45}^4$$
$$+43s_{12}s_{15}^3s_{23}s_{34}s_{45}^4-1228\epsilon s_{12}^2s_{15}^2s_{23}s_{34}s_{45}^4-307s_{12}^2s_{15}^2s_{23}s_{34}s_{45}^4$$
$$-300\epsilon s_{12}^3s_{15}s_{23}s_{34}s_{45}^4-39s_{12}^3s_{15}s_{23}s_{34}s_{45}^4+120\epsilon s_{12}^2s_{34}^5s_{45}^3$$
$$+30s_{12}^2s_{34}^5s_{45}^3+384\epsilon s_{23}^2s_{34}^5s_{45}^3+72s_{23}^2s_{34}^5s_{45}^3-184\epsilon s_{12}s_{15}s_{34}^5s_{45}^3$$
$$-46s_{12}s_{15}s_{34}^5s_{45}^3+632\epsilon s_{12}s_{23}s_{34}^5s_{45}^3+134s_{12}s_{23}s_{34}^5s_{45}^3-192\epsilon s_{15}s_{23}s_{34}^5s_{45}^3$$
$$-36s_{15}s_{23}s_{34}^5s_{45}^3+244\epsilon s_{12}^3s_{15}^4s_{45}^3+61s_{12}^3s_{15}^4s_{45}^3+4\epsilon s_{12}^3s_{23}^4s_{45}^3$$
$$+s_{12}^3s_{23}^4s_{45}^3+124\epsilon s_{12}^2s_{15}s_{23}^4s_{45}^3+31s_{12}^2s_{15}s_{23}^4s_{45}^3-120\epsilon s_{12}^3s_{34}^4s_{45}^3$$
$$-30s_{12}^3s_{34}^4s_{45}^3-656\epsilon s_{23}^3s_{34}^4s_{45}^3-140s_{23}^3s_{34}^4s_{45}^3+616\epsilon s_{12}s_{15}^2s_{34}^4s_{45}^3$$
$$+154s_{12}s_{15}^2s_{34}^4s_{45}^3-2256\epsilon s_{12}s_{23}^2s_{34}^4s_{45}^3-444s_{12}s_{23}^2s_{34}^4s_{45}^3$$
$$-288\epsilon s_{15}s_{23}^2s_{34}^4s_{45}^3-48s_{15}s_{23}^2s_{34}^4s_{45}^3-176\epsilon s_{12}^2s_{15}s_{34}^4s_{45}^3$$
$$-44s_{12}^2s_{15}s_{34}^4s_{45}^3-1512\epsilon s_{12}^2s_{23}s_{34}^4s_{45}^3-306s_{12}^2s_{23}s_{34}^4s_{45}^3$$
$$+464\epsilon s_{15}^2s_{23}s_{34}^4s_{45}^3+92s_{15}^2s_{23}s_{34}^4s_{45}^3-504\epsilon s_{12}s_{15}s_{23}s_{34}^4s_{45}^3$$
$$-150s_{12}s_{15}s_{23}s_{34}^4s_{45}^3-56\epsilon s_{12}^4s_{15}^3s_{45}^3-14s_{12}^4s_{15}^3s_{45}^3+52\epsilon s_{12}^4s_{23}^3s_{45}^3$$
$$+13s_{12}^4s_{23}^3s_{45}^3-112\epsilon s_{12}^2s_{15}^2s_{23}^3s_{45}^3-\text{[illegible]}\,s_{12}^2s_{15}^2s_{23}^3s_{45}^3+360\epsilon s_{12}^3s_{15}s_{23}^3s_{45}^3$$

Look at one example, one 5-page long coefficient

$$+ 90s_{12}^3s_{15}s_{23}^3s_{45}^3 + 160\epsilon s_{23}^4s_{34}^3s_{45}^3 + 40s_{23}^4s_{34}^3s_{45}^3 - 680\epsilon s_{12}s_{15}^3s_{34}^3s_{45}^3$$
$$- 170s_{12}s_{15}^3s_{34}^3s_{45}^3 + 1940\epsilon s_{12}s_{23}^3s_{34}^3s_{45}^3 + 437s_{12}s_{23}^3s_{34}^3s_{45}^3$$
$$+ 800\epsilon s_{15}s_{23}^3s_{34}^3s_{45}^3 + 152s_{15}s_{23}^3s_{34}^3s_{45}^3 - 688\epsilon s_{12}^2s_{15}^2s_{34}^3s_{45}^3$$
$$- 172s_{12}^2s_{15}^2s_{34}^3s_{45}^3 + 3184\epsilon s_{12}^2s_{23}^2s_{34}^3s_{45}^3 + 652s_{12}^2s_{23}^2s_{34}^3s_{45}^3$$
$$- 320\epsilon s_{15}^2s_{23}^2s_{34}^3s_{45}^3 - 56s_{15}^2s_{23}^2s_{34}^3s_{45}^3 + 2480\epsilon s_{12}s_{15}s_{23}^2s_{34}^3s_{45}^3$$
$$+ 452s_{12}s_{15}s_{23}^2s_{34}^3s_{45}^3 + 544\epsilon s_{12}^3s_{15}s_{34}^3s_{45}^3 + 136s_{12}^3s_{15}s_{34}^3s_{45}^3$$
$$+ 1032\epsilon s_{12}^3s_{23}s_{34}^3s_{45}^3 + 198s_{12}^3s_{23}s_{34}^3s_{45}^3 - 352\epsilon s_{15}^3s_{23}s_{34}^3s_{45}^3$$
$$- 76s_{15}^3s_{23}s_{34}^3s_{45}^3 - 1072\epsilon s_{12}s_{15}^2s_{23}s_{34}^3s_{45}^3 - 160s_{12}s_{15}^2s_{23}s_{34}^3s_{45}^3$$
$$+ 1808\epsilon s_{12}^2s_{15}s_{23}s_{34}^3s_{45}^3 + 428s_{12}^2s_{15}s_{23}s_{34}^3s_{45}^3 + 8\epsilon s_{12}^5s_{23}^2s_{45}^3$$
$$+ 2s_{12}^5s_{23}^2s_{45}^3 - 276\epsilon s_{12}^2s_{15}^3s_{23}^2s_{45}^3 - 69s_{12}^2s_{15}^3s_{23}^2s_{45}^3 - 496\epsilon s_{12}^3s_{15}^2s_{23}^2s_{45}^3$$
$$- 124s_{12}^3s_{15}^2s_{23}^2s_{45}^3 - 32\epsilon s_{12}^4s_{15}s_{23}^2s_{45}^3 - 8s_{12}^4s_{15}s_{23}^2s_{45}^3 + 248\epsilon s_{12}s_{15}^4s_{34}^2s_{45}^3$$
$$+ 62s_{12}s_{15}^4s_{34}^2s_{45}^3 - 348\epsilon s_{12}s_{23}^4s_{34}^2s_{45}^3 - 87s_{12}s_{23}^4s_{34}^2s_{45}^3$$
$$- 160\epsilon s_{15}s_{23}^4s_{34}^2s_{45}^3 - 40s_{15}s_{23}^4s_{34}^2s_{45}^3 + 1420\epsilon s_{12}^2s_{15}^3s_{34}^2s_{45}^3$$
$$+ 355s_{12}^2s_{15}^3s_{34}^2s_{45}^3 - 1688\epsilon s_{12}^2s_{23}^3s_{34}^2s_{45}^3 - 398s_{12}^2s_{23}^3s_{34}^2s_{45}^3$$
$$- 144\epsilon s_{15}^2s_{23}^3s_{34}^2s_{45}^3 - 12s_{15}^2s_{23}^3s_{34}^2s_{45}^3 - 1336\epsilon s_{12}s_{15}s_{23}^3s_{34}^2s_{45}^3$$
$$- 262s_{12}s_{15}s_{23}^3s_{34}^2s_{45}^3 - 348\epsilon s_{12}^3s_{15}^2s_{34}^2s_{45}^3 - 87s_{12}^3s_{15}^2s_{34}^2s_{45}^3$$
$$- 1536\epsilon s_{12}^3s_{23}^2s_{34}^2s_{45}^3 - 336s_{12}^3s_{23}^2s_{34}^2s_{45}^3 + 224\epsilon s_{15}^3s_{23}^2s_{34}^2s_{45}^3$$
$$+ 32s_{15}^3s_{23}^2s_{34}^2s_{45}^3 - 808\epsilon s_{12}s_{15}^2s_{23}^2s_{34}^2s_{45}^3 - 178s_{12}s_{15}^2s_{23}^2s_{34}^2s_{45}^3$$
$$- 2064\epsilon s_{12}^2s_{15}s_{23}^2s_{34}^2s_{45}^3 - 348s_{12}^2s_{15}s_{23}^2s_{34}^2s_{45}^3 - 124\epsilon s_{12}^4s_{15}s_{34}^2s_{45}^3$$
$$- 31s_{12}^4s_{15}s_{34}^2s_{45}^3 - 212\epsilon s_{12}^4s_{23}s_{34}^2s_{45}^3 - 41s_{12}^4s_{23}s_{34}^2s_{45}^3 + 80\epsilon s_{15}^4s_{23}s_{34}^2s_{45}^3$$
$$+ 20s_{15}^4s_{23}s_{34}^2s_{45}^3 + 1428\epsilon s_{12}s_{15}^3s_{23}s_{34}^2s_{45}^3 + 297s_{12}s_{15}^3s_{23}s_{34}^2s_{45}^3$$
$$- 588\epsilon s_{12}^2s_{15}^2s_{23}s_{34}^2s_{45}^3 - 231s_{12}^2s_{15}^2s_{23}s_{34}^2s_{45}^3 - 676\epsilon s_{12}^3s_{15}s_{23}s_{34}^2s_{45}^3$$
$$- 133s_{12}^3s_{15}s_{23}s_{34}^2s_{45}^3 + 264\epsilon s_{12}^2s_{15}^4s_{23}s_{45}^3 + 66s_{12}^2s_{15}^4s_{23}s_{45}^3$$
$$- 112\epsilon s_{12}^3s_{15}^3s_{23}s_{45}^3 - 28s_{12}^3s_{15}^3s_{23}s_{45}^3 + 36\epsilon s_{12}^4s_{15}^2s_{23}s_{45}^3 + 9s_{12}^4s_{15}^2s_{23}s_{45}^3$$
$$- 8\epsilon s_{12}^5s_{15}s_{23}s_{45}^3 - 2s_{12}^5s_{15}s_{23}s_{45}^3 - 676\epsilon s_{12}^2s_{15}^4s_{34}s_{45}^3 - 169s_{12}^2s_{15}^4s_{34}s_{45}^3$$
$$+ 184\epsilon s_{12}^2s_{23}^4s_{34}s_{45}^3 + 46s_{12}^2s_{23}^4s_{34}s_{45}^3 + 68\epsilon s_{12}s_{15}s_{23}^4s_{34}s_{45}^3$$
$$+ 17s_{12}s_{15}s_{23}^4s_{34}s_{45}^3 - 320\epsilon s_{12}^3s_{15}^3s_{34}s_{45}^3 - 80s_{12}^3s_{15}^3s_{34}s_{45}^3$$
$$+ 412\epsilon s_{12}^3s_{23}^3s_{34}s_{45}^3 + 103s_{12}^3s_{23}^3s_{34}s_{45}^3 + 148\epsilon s_{12}s_{15}^2s_{23}^3s_{34}s_{45}^3$$
$$+ 13s_{12}s_{15}^2s_{23}^3s_{34}s_{45}^3 - 116\epsilon s_{12}^2s_{15}s_{23}^3s_{34}s_{45}^3 - 53s_{12}^2s_{15}s_{23}^3s_{34}s_{45}^3$$
$$+ 180\epsilon s_{12}^4s_{15}^2s_{34}s_{45}^3 + 45s_{12}^4s_{15}^2s_{34}s_{45}^3 + 216\epsilon s_{12}^4s_{23}^2s_{34}s_{45}^3 + 54s_{12}^4s_{23}^2s_{34}s_{45}^3$$
$$+ 268\epsilon s_{12}s_{15}^3s_{23}^2s_{34}s_{45}^3 + 91s_{12}s_{15}^3s_{23}^2s_{34}s_{45}^3 + 1072\epsilon s_{12}^2s_{15}^2s_{23}^2s_{34}s_{45}^3$$
$$+ 244s_{12}^2s_{15}^2s_{23}^2s_{34}s_{45}^3 + 72\epsilon s_{12}^3s_{15}s_{23}^2s_{34}s_{45}^3 - 30s_{12}^3s_{15}s_{23}^2s_{34}s_{45}^3$$
$$- 484\epsilon s_{12}s_{15}^4s_{23}s_{34}s_{45}^3 - 121s_{12}s_{15}^4s_{23}s_{34}s_{45}^3 - 160\epsilon s_{12}^2s_{15}^3s_{23}s_{34}s_{45}^3$$
$$- 4s_{12}^2s_{15}^3s_{23}s_{34}s_{45}^3 + 444\epsilon s_{12}^3s_{15}^2s_{23}s_{34}s_{45}^3 + 135s_{12}^3s_{15}^2s_{23}s_{34}s_{45}^3$$
$$- 92\epsilon s_{12}^4s_{15}s_{23}s_{34}s_{45}^3 - 35s_{12}^4s_{15}s_{23}s_{34}s_{45}^3 - 256\epsilon s_{23}^3s_{34}^5s_{45}^2$$
$$- 52s_{23}^3s_{34}^5s_{45}^2 - 792\epsilon s_{12}s_{23}^2s_{34}^5s_{45}^2 - 162s_{12}s_{23}^2s_{34}^5s_{45}^2 + 192\epsilon s_{15}s_{23}^2s_{34}^5s_{45}^2$$
$$+ 36s_{15}s_{23}^2s_{34}^5s_{45}^2 + 120\epsilon s_{12}^2s_{15}s_{34}^5s_{45}^2 + 30s_{12}^2s_{15}s_{34}^5s_{45}^2 - 336\epsilon s_{12}^2s_{23}s_{34}^5s_{45}^2$$
$$- 72s_{12}^2s_{23}s_{34}^5s_{45}^2 + 448\epsilon s_{12}s_{15}s_{23}s_{34}^5s_{45}^2 + 88s_{12}s_{15}s_{23}s_{34}^5s_{45}^2 - 60\epsilon s_{12}^4s_{15}^4s_{45}^2$$
$$- 15s_{12}^4s_{15}^4s_{45}^2 + 8\epsilon s_{12}^4s_{23}^4s_{45}^2 + 2s_{12}^4s_{23}^4s_{45}^2 - 52\epsilon s_{12}^3s_{15}s_{23}^4s_{45}^2$$
$$- 13s_{12}^3s_{15}s_{23}^4s_{45}^2 + 176\epsilon s_{23}^4s_{34}^4s_{45}^2 + 44s_{23}^4s_{34}^4s_{45}^2 + 1520\epsilon s_{12}s_{23}^3s_{34}^4s_{45}^2$$
$$+ 332s_{12}s_{23}^3s_{34}^4s_{45}^2 + 208\epsilon s_{15}s_{23}^3s_{34}^4s_{45}^2 + 28s_{15}s_{23}^3s_{34}^4s_{45}^2$$
$$- 544\epsilon s_{12}^2s_{15}^2s_{34}^4s_{45}^2 - 136s_{12}^2s_{15}^2s_{34}^4s_{45}^2 + 2144\epsilon s_{12}^2s_{23}^2s_{34}^4s_{45}^2$$
$$+ 440s_{12}^2s_{23}^2s_{34}^4s_{45}^2 - 320\epsilon s_{15}^2s_{23}^2s_{34}^4s_{45}^2 - 56s_{15}^2s_{23}^2s_{34}^4s_{45}^2$$
$$+ 400\epsilon s_{12}s_{15}s_{23}^2s_{34}^4s_{45}^2 + 76s_{12}s_{15}s_{23}^2s_{34}^4s_{45}^2 + 552\epsilon s_{12}^3s_{23}s_{34}^4s_{45}^2$$
$$+ 114s_{12}^3s_{23}s_{34}^4s_{45}^2 - 1064\epsilon s_{12}s_{15}^2s_{23}s_{34}^4s_{45}^2 - 218s_{12}s_{15}^2s_{23}s_{34}^4s_{45}^2$$
$$- 192\epsilon s_{12}^2s_{15}s_{23}s_{34}^4s_{45}^2 - 24\epsilon s_{12}^5s_{23}^3s_{45}^2 - 6s_{12}^5s_{23}^3s_{45}^2 + 12\epsilon s_{12}^3s_{15}^2s_{23}^3s_{45}^2$$

Look at one example, one 5-page long coefficient

$$+3s_{12}^3s_{15}^2s_{23}^3s_{45}^2-132\epsilon s_{12}^4s_{15}s_{23}^3s_{45}^2-33s_{12}^4s_{15}s_{23}^3s_{45}^2-624\epsilon s_{12}s_{23}^4s_{34}^3s_{45}^2$$
$$-156s_{12}s_{23}^4s_{34}^3s_{45}^2-176\epsilon s_{15}s_{23}^4s_{34}^3s_{45}^2-44s_{15}s_{23}^4s_{34}^3s_{45}^2$$
$$+728\epsilon s_{12}^2s_{15}^3s_{34}^3s_{45}^2+182s_{12}^2s_{15}^3s_{34}^3s_{45}^2-2500\epsilon s_{12}^2s_{23}^3s_{34}^3s_{45}^2$$
$$-577s_{12}^2s_{23}^3s_{34}^3s_{45}^2+48\epsilon s_{15}^2s_{23}^3s_{34}^3s_{45}^2+24s_{15}^2s_{23}^3s_{34}^3s_{45}^2$$
$$-1028\epsilon s_{12}s_{15}s_{23}^3s_{34}^3s_{45}^2-173s_{12}s_{15}s_{23}^3s_{34}^3s_{45}^2+428\epsilon s_{12}^3s_{15}^2s_{34}^3s_{45}^2$$
$$+107s_{12}^3s_{15}^2s_{34}^3s_{45}^2-1980\epsilon s_{12}^3s_{23}^2s_{34}^3s_{45}^2-423s_{12}^3s_{23}^2s_{34}^3s_{45}^2$$
$$+128\epsilon s_{15}^3s_{23}^2s_{34}^3s_{45}^2+20s_{15}^3s_{23}^2s_{34}^3s_{45}^2+356\epsilon s_{12}s_{15}^2s_{23}^2s_{34}^3s_{45}^2$$
$$+41s_{12}s_{15}^2s_{23}^2s_{34}^3s_{45}^2-748\epsilon s_{12}^2s_{15}s_{23}^2s_{34}^3s_{45}^2-79s_{12}^2s_{15}s_{23}^2s_{34}^3s_{45}^2$$
$$-120\epsilon s_{12}^4s_{15}s_{34}^3s_{45}^2-30s_{12}^4s_{15}s_{34}^3s_{45}^2-216\epsilon s_{12}^4s_{23}s_{34}^3s_{45}^2-42s_{12}^4s_{23}s_{34}^3s_{45}^2$$
$$+848\epsilon s_{12}s_{15}^3s_{23}s_{34}^3s_{45}^2+188s_{12}s_{15}^3s_{23}s_{34}^3s_{45}^2+876\epsilon s_{12}^2s_{15}^2s_{23}s_{34}^3s_{45}^2$$
$$+111s_{12}^2s_{15}^2s_{23}s_{34}^3s_{45}^2+20\epsilon s_{12}^3s_{15}s_{23}s_{34}^3s_{45}^2-19s_{12}^3s_{15}s_{23}s_{34}^3s_{45}^2$$
$$+132\epsilon s_{12}^3s_{15}^3s_{23}^2s_{45}^2+33s_{12}^3s_{15}^3s_{23}^2s_{45}^2+180\epsilon s_{12}^4s_{15}^2s_{23}^2s_{45}^2$$
$$+45s_{12}^4s_{15}^2s_{23}^2s_{45}^2+48\epsilon s_{12}^5s_{15}s_{23}^2s_{45}^2+12s_{12}^5s_{15}s_{23}^2s_{45}^2$$
$$-304\epsilon s_{12}^2s_{15}^4s_{34}^2s_{45}^2-76s_{12}^2s_{15}^4s_{34}^2s_{45}^2+668\epsilon s_{12}^2s_{23}^4s_{34}^2s_{45}^2$$
$$+167s_{12}^2s_{23}^4s_{34}^2s_{45}^2+388\epsilon s_{12}s_{15}s_{23}^4s_{34}^2s_{45}^2+97s_{12}s_{15}s_{23}^4s_{34}^2s_{45}^2$$
$$-792\epsilon s_{12}^3s_{15}^3s_{34}^2s_{45}^2-198s_{12}^3s_{15}^3s_{34}^2s_{45}^2+1536\epsilon s_{12}^3s_{23}^3s_{34}^2s_{45}^2$$
$$+372s_{12}^3s_{23}^3s_{34}^2s_{45}^2-128\epsilon s_{12}s_{15}^2s_{23}^3s_{34}^2s_{45}^2-68s_{12}s_{15}^2s_{23}^3s_{34}^2s_{45}^2$$
$$+528\epsilon s_{12}^2s_{15}s_{23}^3s_{34}^2s_{45}^2+72s_{12}^2s_{15}s_{23}^3s_{34}^2s_{45}^2+56\epsilon s_{12}^4s_{15}^2s_{34}^2s_{45}^2$$
$$+14s_{12}^4s_{15}^2s_{34}^2s_{45}^2+732\epsilon s_{12}^4s_{23}^2s_{34}^2s_{45}^2+171s_{12}^4s_{23}^2s_{34}^2s_{45}^2$$
$$-28\epsilon s_{12}s_{15}^3s_{23}^2s_{34}^2s_{45}^2+29s_{12}s_{15}^3s_{23}^2s_{34}^2s_{45}^2+224\epsilon s_{12}^2s_{15}^2s_{23}^2s_{34}^2s_{45}^2$$
$$+56s_{12}^2s_{15}^2s_{23}^2s_{34}^2s_{45}^2-240\epsilon s_{12}^3s_{15}s_{23}^2s_{34}^2s_{45}^2-132s_{12}^3s_{15}s_{23}^2s_{34}^2s_{45}^2$$
$$-232\epsilon s_{12}s_{15}^4s_{23}s_{34}^2s_{45}^2-58s_{12}s_{15}^4s_{23}s_{34}^2s_{45}^2-828\epsilon s_{12}^2s_{15}^3s_{23}s_{34}^2s_{45}^2$$
$$-159s_{12}^2s_{15}^3s_{23}s_{34}^2s_{45}^2+192\epsilon s_{12}^3s_{15}^2s_{23}s_{34}^2s_{45}^2+108s_{12}^3s_{15}^2s_{23}s_{34}^2s_{45}^2$$
$$-380\epsilon s_{12}^4s_{15}s_{23}s_{34}^2s_{45}^2-95s_{12}^4s_{15}s_{23}s_{34}^2s_{45}^2-92\epsilon s_{12}^3s_{15}^4s_{23}s_{45}^2$$
$$-23s_{12}^3s_{15}^4s_{23}s_{45}^2+4\epsilon s_{12}^4s_{15}^3s_{23}s_{45}^2+s_{12}^4s_{15}^3s_{23}s_{45}^2-24\epsilon s_{12}^5s_{15}^2s_{23}s_{45}^2$$
$$-6s_{12}^5s_{15}^2s_{23}s_{45}^2+364\epsilon s_{12}^3s_{15}^4s_{34}s_{45}^2+91s_{12}^3s_{15}^4s_{34}s_{45}^2-228\epsilon s_{12}^3s_{23}^4s_{34}s_{45}^2$$
$$-57s_{12}^3s_{23}^4s_{34}s_{45}^2-40\epsilon s_{12}^2s_{15}s_{23}^4s_{34}s_{45}^2-10s_{12}^2s_{15}s_{23}^4s_{34}s_{45}^2$$
$$+124\epsilon s_{12}^4s_{15}^3s_{34}s_{45}^2+31s_{12}^4s_{15}^3s_{34}s_{45}^2-276\epsilon s_{12}^4s_{23}^3s_{34}s_{45}^2-69s_{12}^4s_{23}^3s_{34}s_{45}^2$$
$$+160\epsilon s_{12}^2s_{15}^2s_{23}^3s_{34}s_{45}^2+52s_{12}^2s_{15}^2s_{23}^3s_{34}s_{45}^2+448\epsilon s_{12}^3s_{15}s_{23}^3s_{34}s_{45}^2$$
$$+124s_{12}^3s_{15}s_{23}^3s_{34}s_{45}^2-104\epsilon s_{12}^5s_{23}^2s_{34}s_{45}^2-26s_{12}^5s_{23}^2s_{34}s_{45}^2$$
$$-536\epsilon s_{12}^2s_{15}^3s_{23}^2s_{34}s_{45}^2-146s_{12}^2s_{15}^3s_{23}^2s_{34}s_{45}^2-504\epsilon s_{12}^3s_{15}^2s_{23}^2s_{34}s_{45}^2$$
$$-126s_{12}^3s_{15}^2s_{23}^2s_{34}s_{45}^2+252\epsilon s_{12}^4s_{15}s_{23}^2s_{34}s_{45}^2+75s_{12}^4s_{15}s_{23}^2s_{34}s_{45}^2$$
$$+416\epsilon s_{12}^2s_{15}^4s_{23}s_{34}s_{45}^2+104s_{12}^2s_{15}^4s_{23}s_{34}s_{45}^2-80\epsilon s_{12}^3s_{15}^3s_{23}s_{34}s_{45}^2$$
$$-32s_{12}^3s_{15}^3s_{23}s_{34}s_{45}^2-100\epsilon s_{12}^4s_{15}^2s_{23}s_{34}s_{45}^2-37s_{12}^4s_{15}^2s_{23}s_{34}s_{45}^2$$
$$+104\epsilon s_{12}^5s_{15}s_{23}s_{34}s_{45}^2+26s_{12}^5s_{15}s_{23}s_{34}s_{45}^2+64\epsilon s_{23}^4s_{34}^5s_{45}+16s_{23}^4s_{34}^5s_{45}$$
$$+440\epsilon s_{12}s_{23}^3s_{34}^5s_{45}+98s_{12}s_{23}^3s_{34}^5s_{45}-64\epsilon s_{15}s_{23}^3s_{34}^5s_{45}-16s_{15}s_{23}^3s_{34}^5s_{45}$$
$$+312\epsilon s_{12}^2s_{23}^2s_{34}^5s_{45}+66s_{12}^2s_{23}^2s_{34}^5s_{45}-344\epsilon s_{12}s_{15}s_{23}^2s_{34}^5s_{45}$$
$$-74s_{12}s_{15}s_{23}^2s_{34}^5s_{45}-216\epsilon s_{12}^2s_{15}s_{23}s_{34}^5s_{45}-42s_{12}^2s_{15}s_{23}s_{34}^5s_{45}$$
$$-408\epsilon s_{12}s_{23}^4s_{34}^4s_{45}-102s_{12}s_{23}^4s_{34}^4s_{45}-64\epsilon s_{15}s_{23}^4s_{34}^4s_{45}-16s_{15}s_{23}^4s_{34}^4s_{45}$$
$$-1288\epsilon s_{12}^2s_{23}^3s_{34}^4s_{45}-298s_{12}^2s_{23}^3s_{34}^4s_{45}+64\epsilon s_{15}^2s_{23}^3s_{34}^4s_{45}$$
$$+16s_{15}^2s_{23}^3s_{34}^4s_{45}-152\epsilon s_{12}s_{15}s_{23}^3s_{34}^4s_{45}-14s_{12}s_{15}s_{23}^3s_{34}^4s_{45}$$
$$+120\epsilon s_{12}^3s_{15}^2s_{34}^4s_{45}+30s_{12}^3s_{15}^2s_{34}^4s_{45}-720\epsilon s_{12}^3s_{23}^2s_{34}^4s_{45}$$
$$-156s_{12}^3s_{23}^2s_{34}^4s_{45}+464\epsilon s_{12}s_{15}^2s_{23}^2s_{34}^4s_{45}+92s_{12}s_{15}^2s_{23}^2s_{34}^4s_{45}$$
$$+576\epsilon s_{12}^2s_{15}s_{23}^2s_{34}^4s_{45}+144s_{12}^2s_{15}s_{23}^2s_{34}^4s_{45}+424\epsilon s_{12}^2s_{15}^2s_{23}s_{34}^4s_{45}$$
$$+82s_{12}^2s_{15}^2s_{23}s_{34}^4s_{45}+408\epsilon s_{12}^3s_{15}s_{23}s_{34}^4s_{45}+78s_{12}^3s_{15}s_{23}s_{34}^4s_{45}$$
$$+744\epsilon s_{12}^2s_{23}^4s_{34}^3s_{45}+186s_{12}^2s_{23}^4s_{34}^3s_{45}+344\epsilon s_{12}s_{15}s_{23}^4s_{34}^3s_{45}$$

Look at one example, one 5-page long coefficient

$$+ 86 s_{12} s_{15} s_{23}^4 s_{34}^3 s_{45} - 240\epsilon s_{12}^3 s_{15}^3 s_{34}^3 s_{45} - 60 s_{12}^3 s_{15}^3 s_{34}^3 s_{45}$$
$$+ 1344\epsilon s_{12}^3 s_{23}^3 s_{34}^3 s_{45} + 324 s_{12}^3 s_{23}^3 s_{34}^3 s_{45} - 224\epsilon s_{12} s_{15}^2 s_{23}^3 s_{34}^3 s_{45}$$
$$- 68 s_{12} s_{15}^2 s_{23}^3 s_{34}^3 s_{45} + 36\epsilon s_{12}^2 s_{15} s_{23}^3 s_{34}^3 s_{45} - 27 s_{12}^2 s_{15} s_{23}^3 s_{34}^3 s_{45}$$
$$- 120\epsilon s_{12}^4 s_{15}^2 s_{34}^3 s_{45} - 30 s_{12}^4 s_{15}^2 s_{34}^3 s_{45} + 504\epsilon s_{12}^4 s_{23}^2 s_{34}^3 s_{45}$$
$$+ 114 s_{12}^4 s_{23}^2 s_{34}^3 s_{45} - 120\epsilon s_{12} s_{15}^3 s_{23}^2 s_{34}^3 s_{45} - 18 s_{12} s_{15}^3 s_{23}^2 s_{34}^3 s_{45}$$
$$- 324\epsilon s_{12}^2 s_{15}^2 s_{23}^2 s_{34}^3 s_{45} - 57 s_{12}^2 s_{15}^2 s_{23}^2 s_{34}^3 s_{45} - 788\epsilon s_{12}^3 s_{15} s_{23}^2 s_{34}^3 s_{45}$$
$$- 221 s_{12}^3 s_{15} s_{23}^2 s_{34}^3 s_{45} - 360\epsilon s_{12}^2 s_{15}^3 s_{23} s_{34}^3 s_{45} - 78 s_{12}^2 s_{15}^3 s_{23} s_{34}^3 s_{45}$$
$$- 124\epsilon s_{12}^3 s_{15}^2 s_{23} s_{34}^3 s_{45} + 5 s_{12}^3 s_{15}^2 s_{23} s_{34}^3 s_{45} - 288\epsilon s_{12}^4 s_{15} s_{23} s_{34}^3 s_{45}$$
$$- 60 s_{12}^4 s_{15} s_{23} s_{34}^3 s_{45} + 120\epsilon s_{12}^3 s_{15}^4 s_{34}^2 s_{45} + 30 s_{12}^3 s_{15}^4 s_{34}^2 s_{45}$$
$$- 520\epsilon s_{12}^3 s_{23}^4 s_{34}^2 s_{45} - 130 s_{12}^3 s_{23}^4 s_{34}^2 s_{45} - 340\epsilon s_{12}^2 s_{15} s_{23}^4 s_{34}^2 s_{45}$$
$$- 85 s_{12}^2 s_{15} s_{23}^4 s_{34}^2 s_{45} + 180\epsilon s_{12}^4 s_{15}^3 s_{34}^2 s_{45} + 45 s_{12}^4 s_{15}^3 s_{34}^2 s_{45}$$
$$- 584\epsilon s_{12}^4 s_{23}^3 s_{34}^2 s_{45} - 146 s_{12}^4 s_{23}^3 s_{34}^2 s_{45} + 528\epsilon s_{12}^2 s_{15}^2 s_{23}^3 s_{34}^2 s_{45}$$
$$+ 144 s_{12}^2 s_{15}^2 s_{23}^3 s_{34}^2 s_{45} + 424\epsilon s_{12}^3 s_{15} s_{23}^3 s_{34}^2 s_{45} + 118 s_{12}^3 s_{15} s_{23}^3 s_{34}^2 s_{45}$$
$$- 96\epsilon s_{12}^5 s_{23}^2 s_{34}^2 s_{45} - 24 s_{12}^5 s_{23}^2 s_{34}^2 s_{45} - 340\epsilon s_{12}^2 s_{15}^3 s_{23}^2 s_{34}^2 s_{45}$$
$$- 97 s_{12}^2 s_{15}^3 s_{23}^2 s_{34}^2 s_{45} + 8\epsilon s_{12}^3 s_{15}^2 s_{23}^2 s_{34}^2 s_{45} + 2 s_{12}^3 s_{15}^2 s_{23}^2 s_{34}^2 s_{45}$$
$$+ 732\epsilon s_{12}^4 s_{15} s_{23}^2 s_{34}^2 s_{45} + 195 s_{12}^4 s_{15} s_{23}^2 s_{34}^2 s_{45} + 152\epsilon s_{12}^2 s_{15}^4 s_{23} s_{34}^2 s_{45}$$
$$+ 38 s_{12}^2 s_{15}^4 s_{23} s_{34}^2 s_{45} - 32\epsilon s_{12}^3 s_{15}^3 s_{23} s_{34}^2 s_{45} - 20 s_{12}^3 s_{15}^3 s_{23} s_{34}^2 s_{45}$$
$$- 328\epsilon s_{12}^4 s_{15}^2 s_{23} s_{34}^2 s_{45} - 94 s_{12}^4 s_{15}^2 s_{23} s_{34}^2 s_{45} + 96\epsilon s_{12}^5 s_{15} s_{23} s_{34}^2 s_{45}$$
$$+ 24 s_{12}^5 s_{15} s_{23} s_{34}^2 s_{45} - 60\epsilon s_{12}^4 s_{15}^4 s_{34} s_{45} - 15 s_{12}^4 s_{15}^4 s_{34} s_{45} + 120\epsilon s_{12}^4 s_{23}^4 s_{34} s_{45}$$
$$+ 30 s_{12}^4 s_{23}^4 s_{34} s_{45} + 60\epsilon s_{12}^3 s_{15} s_{23}^4 s_{34} s_{45} + 15 s_{12}^3 s_{15} s_{23}^4 s_{34} s_{45} + 88\epsilon s_{12}^5 s_{23}^3 s_{34} s_{45}$$
$$+ 22 s_{12}^5 s_{23}^3 s_{34} s_{45} - 212\epsilon s_{12}^3 s_{15}^2 s_{23}^3 s_{34} s_{45} - 53 s_{12}^3 s_{15}^2 s_{23}^3 s_{34} s_{45}$$
$$- 244\epsilon s_{12}^4 s_{15} s_{23}^3 s_{34} s_{45} - 61 s_{12}^4 s_{15} s_{23}^3 s_{34} s_{45} + 244\epsilon s_{12}^3 s_{15}^3 s_{23}^2 s_{34} s_{45}$$
$$+ 61 s_{12}^3 s_{15}^3 s_{23}^2 s_{34} s_{45} + 68\epsilon s_{12}^4 s_{15}^2 s_{23}^2 s_{34} s_{45} + 17 s_{12}^4 s_{15}^2 s_{23}^2 s_{34} s_{45}$$
$$- 176\epsilon s_{12}^5 s_{15} s_{23}^2 s_{34} s_{45} - 44 s_{12}^5 s_{15} s_{23}^2 s_{34} s_{45} - 92\epsilon s_{12}^3 s_{15}^4 s_{23} s_{34} s_{45}$$
$$- 23 s_{12}^3 s_{15}^4 s_{23} s_{34} s_{45} + 116\epsilon s_{12}^4 s_{15}^3 s_{23} s_{34} s_{45} + 29 s_{12}^4 s_{15}^3 s_{23} s_{34} s_{45}$$
$$+ 88\epsilon s_{12}^5 s_{15}^2 s_{23} s_{34} s_{45} + 22 s_{12}^5 s_{15}^2 s_{23} s_{34} s_{45} - 96\epsilon s_{12} s_{23}^4 s_{34}^5 - 24 s_{12} s_{23}^4 s_{34}^5$$
$$- 96\epsilon s_{12}^2 s_{23}^3 s_{34}^5 - 24 s_{12}^2 s_{23}^3 s_{34}^5 + 96\epsilon s_{12} s_{15} s_{23}^3 s_{34}^5 + 24 s_{12} s_{15} s_{23}^3 s_{34}^5$$
$$+ 96\epsilon s_{12}^2 s_{15} s_{23}^2 s_{34}^5 + 24 s_{12}^2 s_{15} s_{23}^2 s_{34}^5 + 288\epsilon s_{12}^2 s_{23}^4 s_{34}^4 + 72 s_{12}^2 s_{23}^4 s_{34}^4$$
$$+ 96\epsilon s_{12} s_{15} s_{23}^4 s_{34}^4 + 24 s_{12} s_{15} s_{23}^4 s_{34}^4 + 288\epsilon s_{12}^3 s_{23}^3 s_{34}^4 + 72 s_{12}^3 s_{23}^3 s_{34}^4$$
$$- 96\epsilon s_{12} s_{15}^2 s_{23}^3 s_{34}^4 - 24 s_{12} s_{15}^2 s_{23}^3 s_{34}^4 - 288\epsilon s_{12}^2 s_{15} s_{23}^3 s_{34}^4 - 72 s_{12}^2 s_{15} s_{23}^3 s_{34}^4$$
$$- 384\epsilon s_{12}^3 s_{15} s_{23}^2 s_{34}^4 - 96 s_{12}^3 s_{15} s_{23}^2 s_{34}^4 + 96\epsilon s_{12}^3 s_{15}^2 s_{23} s_{34}^4 + 24 s_{12}^3 s_{15}^2 s_{23} s_{34}^4$$
$$- 288\epsilon s_{12}^3 s_{23}^4 s_{34}^3 - 72 s_{12}^3 s_{23}^4 s_{34}^3 - 192\epsilon s_{12}^2 s_{15} s_{23}^4 s_{34}^3 - 48 s_{12}^2 s_{15} s_{23}^4 s_{34}^3$$
$$- 288\epsilon s_{12}^4 s_{23}^3 s_{34}^3 - 72 s_{12}^4 s_{23}^3 s_{34}^3 + 288\epsilon s_{12}^2 s_{15}^2 s_{23}^3 s_{34}^3 + 72 s_{12}^2 s_{15}^2 s_{23}^3 s_{34}^3$$
$$+ 288\epsilon s_{12}^3 s_{15} s_{23}^3 s_{34}^3 + 72 s_{12}^3 s_{15} s_{23}^3 s_{34}^3 - 96\epsilon s_{12}^2 s_{15}^3 s_{23}^2 s_{34}^3$$
$$- 24 s_{12}^2 s_{15}^3 s_{23}^2 s_{34}^3 + 96\epsilon s_{12}^3 s_{15}^2 s_{23}^2 s_{34}^3 + 24 s_{12}^3 s_{15}^2 s_{23}^2 s_{34}^3$$
$$+ 480\epsilon s_{12}^4 s_{15} s_{23}^2 s_{34}^3 + 120 s_{12}^4 s_{15} s_{23}^2 s_{34}^3 - 96\epsilon s_{12}^3 s_{15}^3 s_{23} s_{34}^3$$
$$- 24 s_{12}^3 s_{15}^3 s_{23} s_{34}^3 - 192\epsilon s_{12}^4 s_{15}^2 s_{23} s_{34}^3 - 48 s_{12}^4 s_{15}^2 s_{23} s_{34}^3 + 96\epsilon s_{12}^4 s_{23}^4 s_{34}^2$$
$$+ 24 s_{12}^4 s_{23}^4 s_{34}^2 + 96\epsilon s_{12}^3 s_{15} s_{23}^4 s_{34}^2 + 24 s_{12}^3 s_{15} s_{23}^4 s_{34}^2 + 96\epsilon s_{12}^5 s_{23}^3 s_{34}^2$$
$$+ 24 s_{12}^5 s_{23}^3 s_{34}^2 - 192\epsilon s_{12}^3 s_{15}^2 s_{23}^3 s_{34}^2 - 48 s_{12}^3 s_{15}^2 s_{23}^3 s_{34}^2 - 96\epsilon s_{12}^4 s_{15} s_{23}^3 s_{34}^2$$
$$- 24 s_{12}^4 s_{15} s_{23}^3 s_{34}^2 + 96\epsilon s_{12}^3 s_{15}^3 s_{23}^2 s_{34}^2 + 24 s_{12}^3 s_{15}^3 s_{23}^2 s_{34}^2$$
$$- 96\epsilon s_{12}^4 s_{15}^2 s_{23}^2 s_{34}^2 - 24 s_{12}^4 s_{15}^2 s_{23}^2 s_{34}^2 - 192\epsilon s_{12}^5 s_{15} s_{23}^2 s_{34}^2$$
$$- 48 s_{12}^5 s_{15} s_{23}^2 s_{34}^2 + 96\epsilon s_{12}^4 s_{15}^3 s_{23} s_{34}^2 + 24 s_{12}^4 s_{15}^3 s_{23} s_{34}^2 + 96\epsilon s_{12}^5 s_{15}^2 s_{23} s_{34}^2$$
$$+ 24 s_{12}^5 s_{15}^2 s_{23} s_{34}^2)/(8(4\epsilon+1) s_{12} s_{23} s_{45}^2 (s_{12}-s_{45})(s_{34}+s_{45})(s_{12}+s_{15}-s_{34})(s_{12}+s_{23}-s_{45})$$
$$(s_{12}-s_{34}-s_{45})(-s_{15}+s_{23}+s_{34})(-s_{15}+s_{23}-s_{45}))$$

page 5

Using our algorithm,
this coefficient is simplified to

$$
\begin{aligned}
= \; & \frac{3s_{23}s_{34}}{2(4\epsilon+1)s_{12}s_{45}(-s_{15}+s_{23}+s_{34})} - \frac{3s_{34}}{2(4\epsilon+1)s_{12}s_{45}} + \frac{15{s_{15}}^2 - 15s_{15}s_{34}}{8s_{23}s_{45}(-s_{12}-s_{15}+s_{34})} \\
& + \frac{s_{23}{s_{34}}^2}{{s_{45}}^2(s_{45}-s_{12})(-s_{15}+s_{23}+s_{34})} - \frac{2s_{23}{s_{34}}^2}{s_{12}{s_{45}}^2(-s_{15}+s_{23}+s_{34})} + \frac{s_{23}s_{34}+{s_{34}}^2}{s_{45}(s_{45}-s_{12})(-s_{15}+s_{23}+s_{34})} \\
& - \frac{11s_{23}s_{34}}{2s_{12}s_{45}(-s_{15}+s_{23}+s_{34})} - \frac{15s_{15}s_{34}}{8s_{23}s_{45}(-s_{12}+s_{34}+s_{45})} + \frac{s_{15}-s_{23}-s_{34}}{s_{45}(-s_{12}-s_{23}+s_{45})} + \frac{2s_{15}-2s_{34}}{s_{12}s_{23}} \\
& + \frac{15s_{15}-15s_{34}}{8s_{23}(-s_{12}-s_{15}+s_{34})} - \frac{7s_{23}}{2s_{12}(-s_{15}+s_{23}+s_{34})} - \frac{15s_{23}}{4(-s_{12}-s_{15}+s_{34})(-s_{15}+s_{23}+s_{34})} \\
& - \frac{s_{15}}{2s_{12}(-s_{12}-s_{23}+s_{45})} + \frac{s_{23}-s_{45}}{2s_{12}(s_{15}-s_{23}+s_{45})} + \frac{15s_{15}}{8s_{45}(-s_{12}-s_{15}+s_{34})} \\
& + \frac{15}{4(-s_{12}-s_{15}+s_{34})} + \frac{7s_{34}}{4s_{23}(-s_{12}-s_{23}+s_{45})} - \frac{5s_{34}}{4(s_{45}-s_{12})(-s_{12}-s_{23}+s_{45})} \\
& - \frac{15s_{34}}{8s_{23}(-s_{12}+s_{34}+s_{45})} + \frac{1}{2(-s_{12}-s_{23}+s_{45})} + \frac{4s_{34}}{s_{12}s_{45}} - \frac{11s_{34}}{4s_{45}(s_{45}-s_{12})} \\
& - \frac{15}{8(-s_{12}+s_{34}+s_{45})} + \frac{5}{4(s_{45}-s_{12})} + \frac{4}{s_{12}} - \frac{3{s_{34}}^2}{{s_{45}}^2(-s_{15}+s_{23}+s_{34})} \\
& + \frac{s_{34}}{4s_{45}(-s_{15}+s_{23}+s_{34})} + \frac{s_{45}}{2(s_{34}+s_{45})(s_{15}-s_{23}+s_{45})} + \frac{3s_{34}}{{s_{45}}^2} - \frac{1}{2(s_{34}+s_{45})} - \frac{1}{4s_{45}}
\end{aligned}
$$

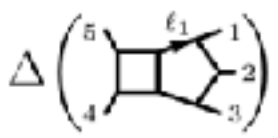
It is ~16 times compression

So other integrals can be expressed as UT integrals with reasonable coefficients

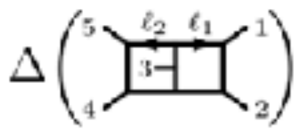
Application

2-loop 5-point +++++ pure-YM amplitude

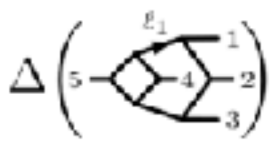
Badger, Frellesvig, **YZ**, 2013
Badger, Mogull, Ochirov, O'Connell 2015



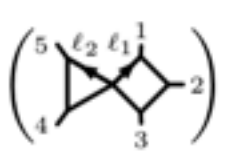
$$\Delta_{431} = \Delta(\cdot) = -\frac{s_{12}s_{23}s_{45}F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}\left(\mathrm{tr}_+(1345)(\ell_1+p_5)^2 + s_{15}s_{34}s_{45}\right),$$



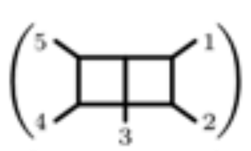
$$\begin{aligned}\Delta_{332} = \Delta(\cdot) = {} & \frac{s_{12}s_{45}F_1}{4\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5} \\ & \times \Big(s_{23}\mathrm{tr}_+(1345)\big(2s_{12} - 4\,\ell_1\cdot(p_5-p_4) + 2(\ell_1-\ell_2)\cdot p_3\big) \\ & - s_{34}\mathrm{tr}_+(1235)\big(2s_{45} - 4\,\ell_2\cdot(p_1-p_2) - 2(\ell_1-\ell_2)\cdot p_3\big) \\ & - 4s_{23}s_{34}s_{15}(\ell_1-\ell_2)\cdot p_3\Big),\end{aligned}$$



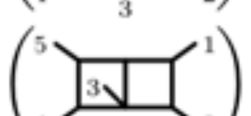
$$\begin{aligned}\Delta_{422} = \Delta(\cdot) = {} & -\frac{s_{12}s_{23}s_{45}F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5} \\ & \times \left(\mathrm{tr}_+(1345)\left(\ell_1\cdot(p_5-p_4) - \frac{s_{45}}{2}\right) + s_{15}s_{34}s_{45}\right).\end{aligned}$$



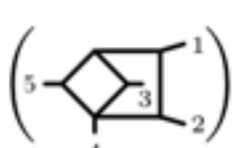
$$\begin{aligned}\Delta_{430} = \Delta(\cdot) = {} & -\frac{s_{12}\mathrm{tr}_+(1345)}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle s_{13}}(2(\ell_1\cdot\omega_{123}) + s_{23}) \\ & \times \left(F_2 + F_3\frac{(\ell_1+\ell_2)^2 + s_{45}}{s_{45}}\right),\end{aligned}$$



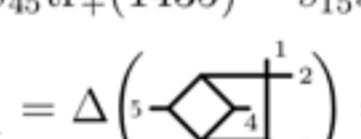
$$\Delta_{331;5L_1} = \Delta(\cdot) = \frac{s_{12}s_{23}s_{34}s_{45}s_{51}F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5},$$

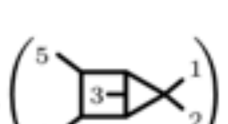


$$\begin{aligned}\Delta_{331;5L_2} = \Delta(\cdot) = {} & -\frac{s_{12}s_{45}F_1}{4\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5} \\ & \times \left(2s_{23}s_{34}s_{15} - s_{23}\mathrm{tr}_+(1345) + s_{34}\mathrm{tr}_+(1235)\right),\end{aligned}$$

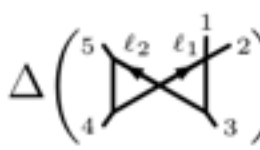


$$\begin{aligned}\Delta_{322;5L_1} = \Delta(\cdot) = {} & -\frac{s_{12}F_1}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5} \\ & \times \left(s_{23}s_{45}\mathrm{tr}_+(1435) - s_{15}s_{34}\mathrm{tr}_+(2453)\right),\end{aligned}$$

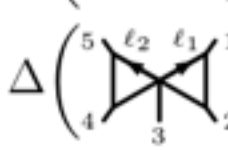




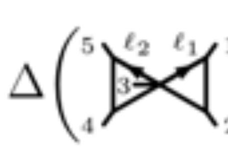
$$\begin{aligned}\Delta_{331;M_1} = \Delta(\cdot) = \Delta_{322;M_1} = \Delta(\cdot) = \Delta_{232;M_1} = \Delta(\cdot) \\ = -\frac{s_{34}s_{45}^2\mathrm{tr}_+(1235)F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5},\end{aligned}$$



$$\Delta_{330;M_1} = \Delta(\cdot) = -\frac{(s_{45}-s_{12})\mathrm{tr}_+(1345)}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle s_{13}}\left(F_2 + F_3\frac{(\ell_1+\ell_2)^2 + s_{45}}{s_{45}}\right),$$



$$\begin{aligned}\Delta_{330;5L_1} = \Delta(\cdot) = {} & -\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \\ & \times \Bigg\{\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right) \\ & \quad \times \left(F_2 + F_3\frac{4(\ell_1\cdot p_3)(\ell_2\cdot p_3) + (\ell_1+\ell_2)^2(s_{12}+s_{45}) + s_{12}s_{45}}{s_{12}s_{45}}\right) \\ & \quad + F_3\Bigg[(\ell_1+\ell_2)^2 s_{15} \\ & \quad + \mathrm{tr}_+(1235)\left(\frac{(\ell_1+\ell_2)^2}{2s_{35}} - \frac{\ell_1\cdot p_3}{s_{12}}\left(1 + \frac{2(\ell_2\cdot\omega_{543})}{s_{35}} + \frac{s_{12}-s_{45}}{s_{35}s_{45}}(\ell_2-p_5)^2\right)\right) \\ & \quad + \mathrm{tr}_+(1345)\left(\frac{(\ell_1+\ell_2)^2}{2s_{13}} - \frac{\ell_2\cdot p_3}{s_{45}}\left(1 + \frac{2(\ell_1\cdot\omega_{123})}{s_{13}} + \frac{s_{45}-s_{12}}{s_{12}s_{13}}(\ell_1-p_1)^2\right)\right)\Bigg]\Bigg\},\end{aligned}$$



$$\begin{aligned}\Delta_{330;5L_2} = \Delta(\cdot) = {} & \frac{F_3}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle s_{12}} \\ & \times \Bigg((s_{45}-s_{12})\mathrm{tr}_+(1245) - \left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)2(\ell_1\cdot p_3) \\ & - \frac{s_{45}\mathrm{tr}_+(1235)}{s_{35}}\left(2(\ell_2\cdot\omega_{543}) + \frac{s_{12}-s_{45}}{s_{45}}(\ell_2-p_5)^2\right) \\ & + \frac{s_{12}\mathrm{tr}_+(1345)}{s_{13}}\left(2(\ell_1\cdot\omega_{123}) + \frac{s_{45}-s_{12}}{s_{12}}(\ell_1-p_1)^2\right)\Bigg).\end{aligned}$$

2-loop 5-point +++++ pure-YM amplitude

numerator degree-5 IBP needed (impossible by current analytic IBP method)
indirect finite-field fitting for the amplitude (after IBP) is applicable

All weight-3, weight-4 part of the amplitude cancels out

$$\mathcal{H}^{(2)} = \sum_{S_5/S_{T_1}} T_1\, \mathcal{H}_1^{(2)} + \sum_{S_5/S_{T_{13}}} T_{13}\, \mathcal{H}_{13}^{(2)}$$

$$I_{123;45} = \mathrm{Li}_2\left(1 - s_{12}/s_{45}\right) + \mathrm{Li}_2\left(1 - s_{23}/s_{45}\right) + \log^2\left(s_{12}/s_{23}\right) + \pi^2/6\,.$$

$$\mathcal{H}_1^{(2,0)} = \sum_{S_{T_1}} \left\{ -\kappa \frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} I_{123;45} + \kappa^2 \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \left[5\, s_{12}s_{23} + s_{12}s_{34} + \frac{\mathrm{tr}_+^2(1245)}{s_{12}s_{45}} \right] \right\},$$

$$\mathcal{H}_{13}^{(2,1)} = \sum_{S_{T_{13}}} \left\{ \kappa \frac{[15]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} \left[I_{234;15} + I_{243;15} - I_{324;15} - 4\, I_{345;12} - 4\, I_{354;12} - 4\, I_{435;12} \right] - 6\,\kappa^2 \left[\frac{s_{23}\,\mathrm{tr}_-(1345)}{s_{34}\,\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} - \frac{3}{2} \frac{[12]^2}{\langle 34\rangle\langle 45\rangle\langle 53\rangle} \right] \right\},$$

Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, **YZ**, Zoia
PhysRevLett. 123 (2019) no.7, 071601

Summary

- Significant progress of analytic Feynman evaluation
- Canonical differential equation: lastest development
- Inspired by D-dimensional cuts/computational algebraic geometry
- Example

All 2-loop five-point massless Feynman integrals

Please also check recents papers
on the analytic Feynman integrals

Vielen Dank!

Infrared structure

(Catani's dipole formula 98)

$$A(s_{ij},\epsilon) = \mathbf{Z}(s_{ij},\epsilon) A^f(s_{ij},\epsilon) \qquad \mathbf{Z}(s_{ij},\epsilon) = \exp g^2 \left(\frac{\mathbf{D}_0}{2\epsilon^2} - \frac{\mathbf{D}}{2\epsilon} \right)$$

$$\mathbf{D}_0 = \sum_{i \neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j \,, \quad \mathbf{D} = \sum_{i \neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j \, \log\left(-\frac{s_{ij}}{\mu^2} \right),$$

$\mathbf{T}_i$ is the adjoint action of $su(N_c)$ Lie algebra.

More references

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