Nuclear Science Computing Center at CCNU

Correlated Dirac eigenvalues & Axial anomaly In Chiral symmetric QCD

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based on arXiv: 2011.04870 & in collaboration with
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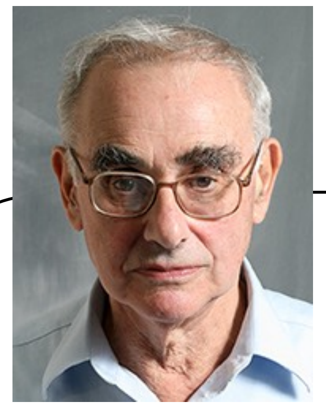
Dec. 10, 2020

Symmetries of QCD in vacuum

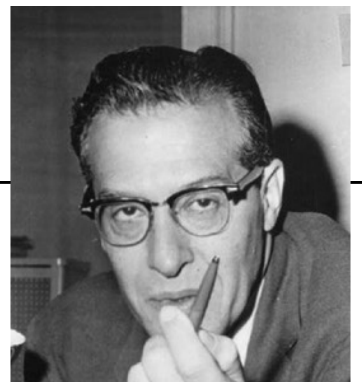
$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_{q=u,d,s,c,b,t} \bar{q}\Big[i\gamma^{\mu}(\partial_{\mu} - igA_{\mu}) - m_q\Big]q$$

Classical QCD symmetry (m_q=0)

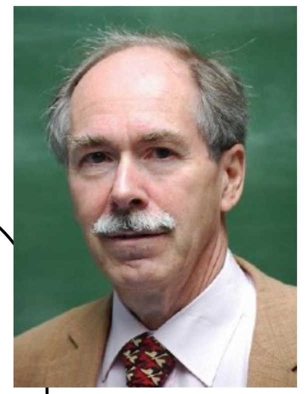
$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$$

Quantum QCD vacuum (m_q=0)





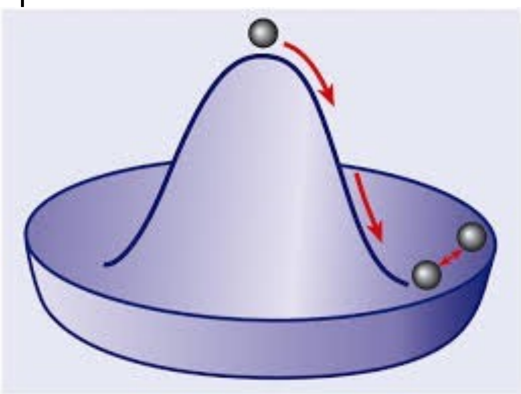


Chiral condensate:
spontaneous mass generation

Axial anomaly:
quantum violation of $U(1)_A$

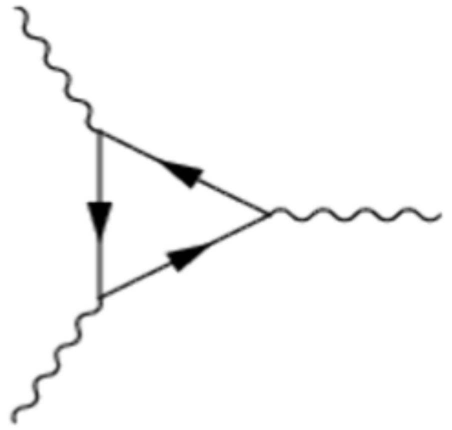
ABJ...

U(1) problem



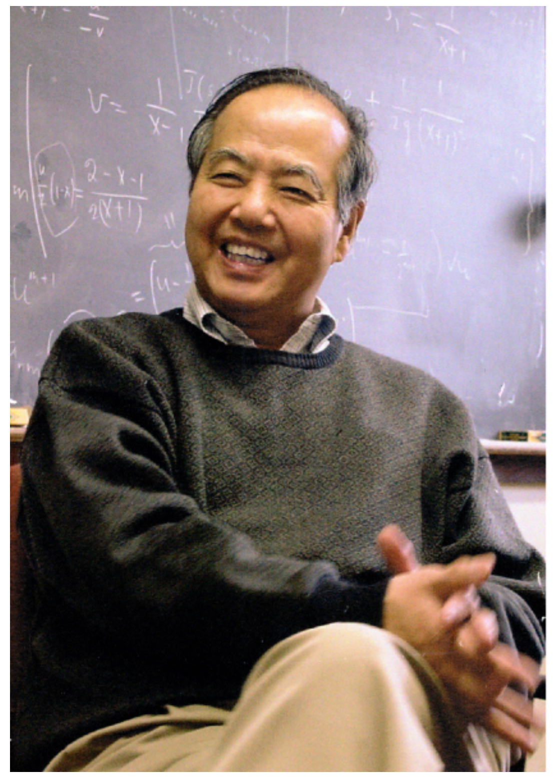
$$\langle \bar{q}_R q_L \rangle \neq 0$$

$$\partial_{\mu} j_5^{\mu} = \frac{g^2 N_f}{16\pi^2} tr(\tilde{F}_{\mu\nu}F^{\mu\nu})$$



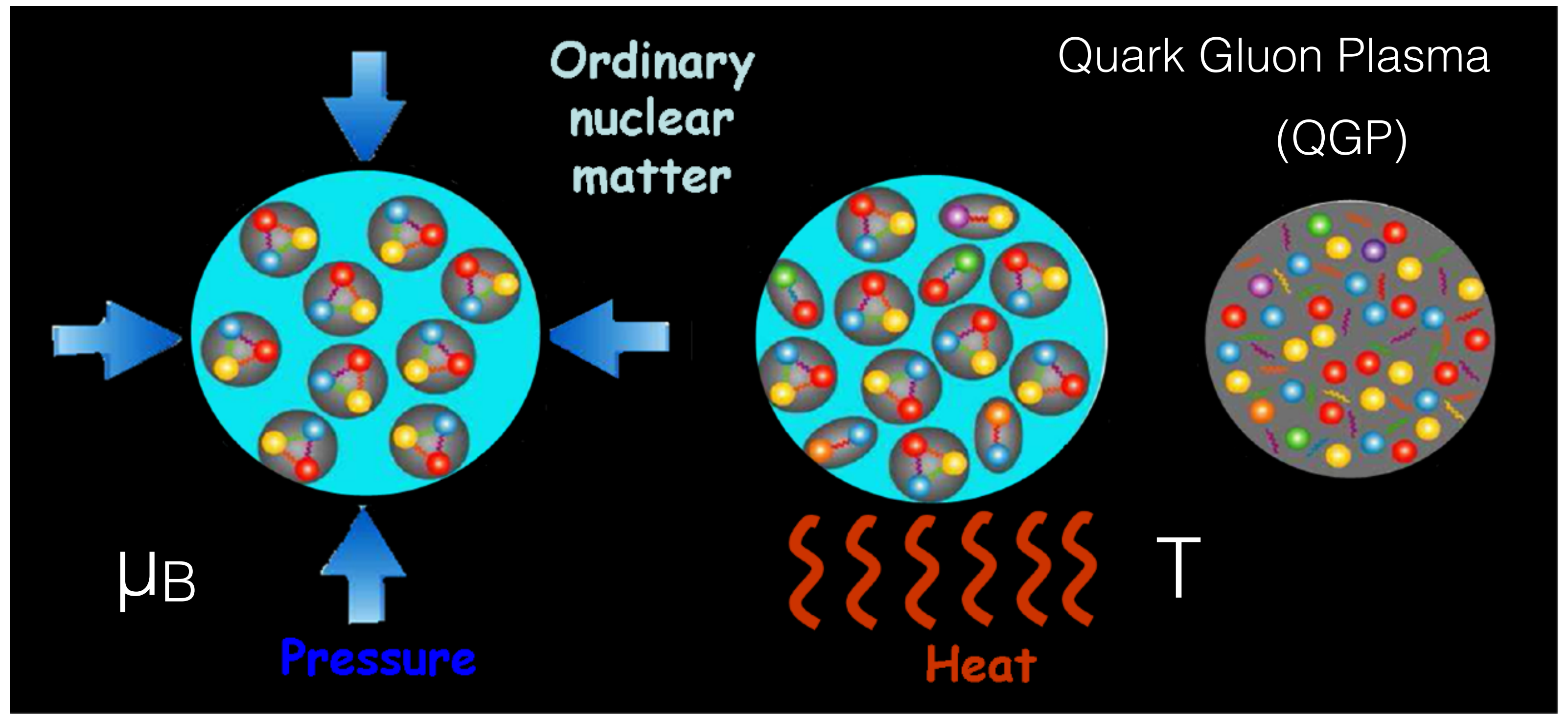
$$SU(N_f)_V \times U(1)_V$$

Missing symmetries & Vacuum excitation



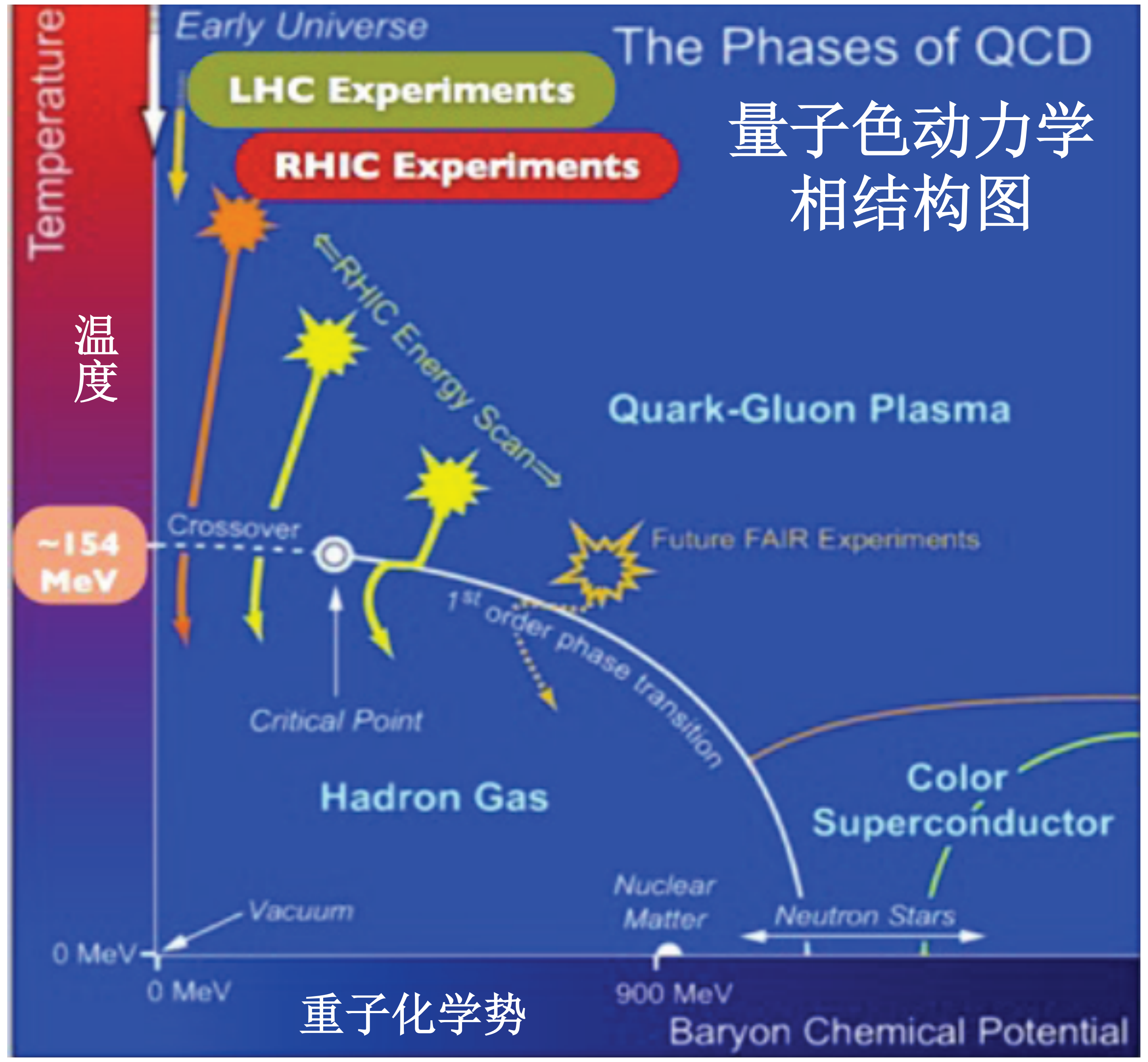
“核子重如牛，对撞生新态。”

Ink painting masterpiece 1986:
"Nuclei as Heavy as Bulls, Through Collision Generate New States of Matter”,
by Li Keran,
reproduced from open source works of
T. D. Lee.



“The whole is more than sum of its parts.”
Aristotle, Metaphysica 10f-1045a

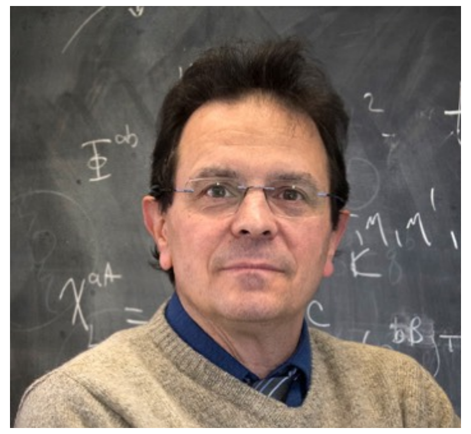
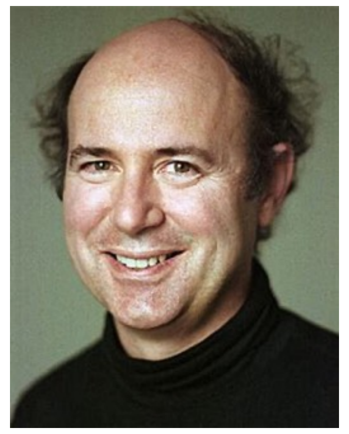
从还原论到整体论



How do symmetries manifest themselves in QCD phase structure?

$$\partial_\mu j_5^\mu = \frac{g^2 N_f}{16\pi^2} tr(\tilde{F}_{\mu\nu} F^{\mu\nu})$$

Landau functional of QCD

Pisarski & Wilczek, PRD 84'

Symmetry: $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$

Chiral field: $\Phi_{ij} \sim \frac{1}{2}\bar{q}^j(1-\gamma_5)q^i = \bar{q}^j_R q^i_L$

Chiral transformation: $\Phi \to e^{-2i\alpha_A} V_L \, \Phi \, V_R^\dagger$

$$\mathcal{L}_{eff} = \frac{1}{2}\,\mathrm{tr}\,\partial\Phi^\dagger\partial\Phi + \frac{a}{2}\,\mathrm{tr}\,\Phi^\dagger\Phi + \frac{b_1}{4!}\left(\mathrm{tr}\,\Phi^\dagger\Phi\right)^2 + \frac{b_2}{4!}\,\mathrm{tr}\,\left(\Phi^\dagger\Phi\right)^2$$

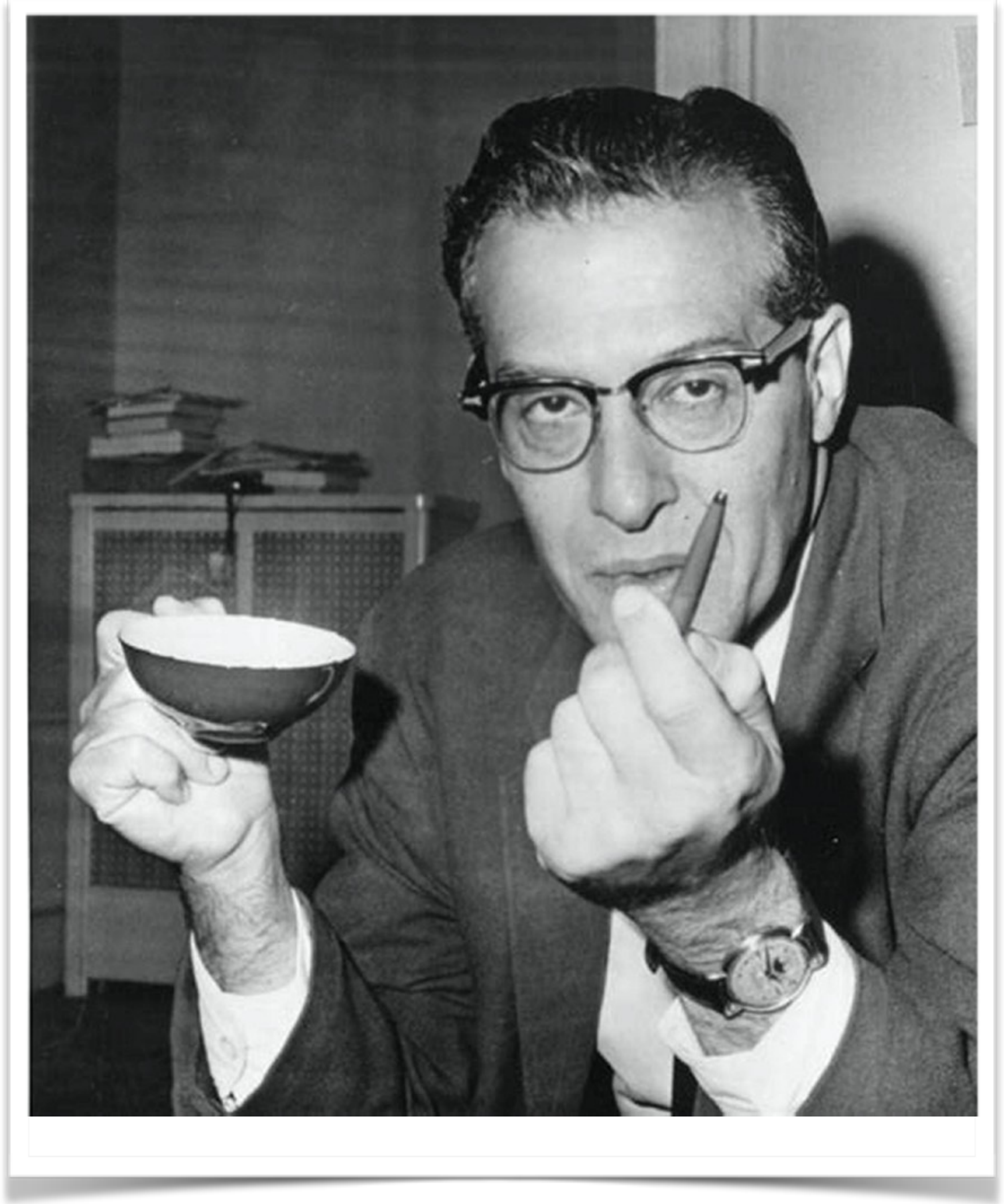
→ $SU(N_f)_L \times SU(N_f)_R \times U(1)_A$

$$- \frac{c}{2}\left(\det\Phi + \det\Phi^\dagger\right)$$

→ $SU(N_f)_L \times SU(N_f)_R$

$$- \frac{d}{2}\mathrm{tr}\, h\left(\Phi + \Phi^\dagger\right).$$

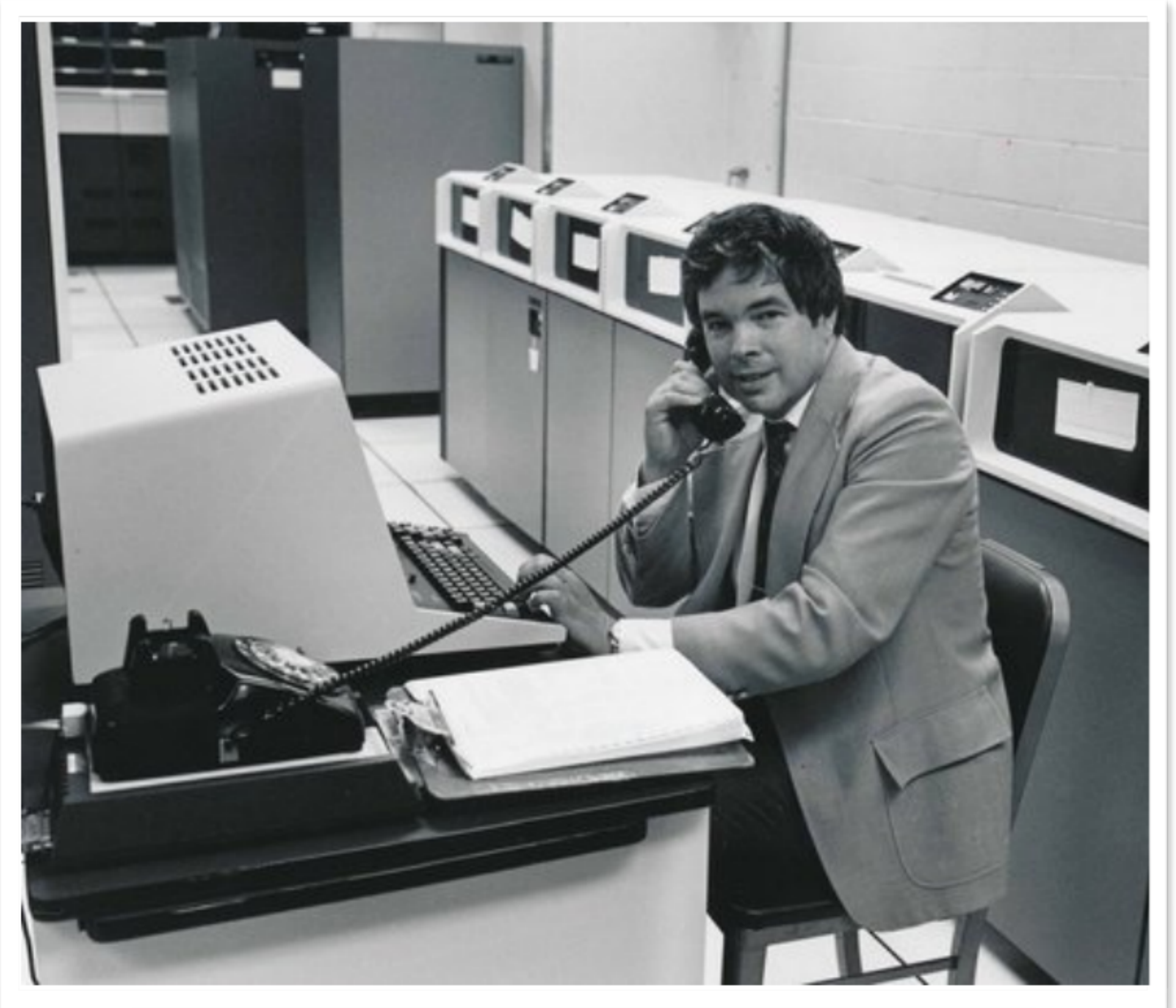
→ Quark mass term



Julian Schwinger:

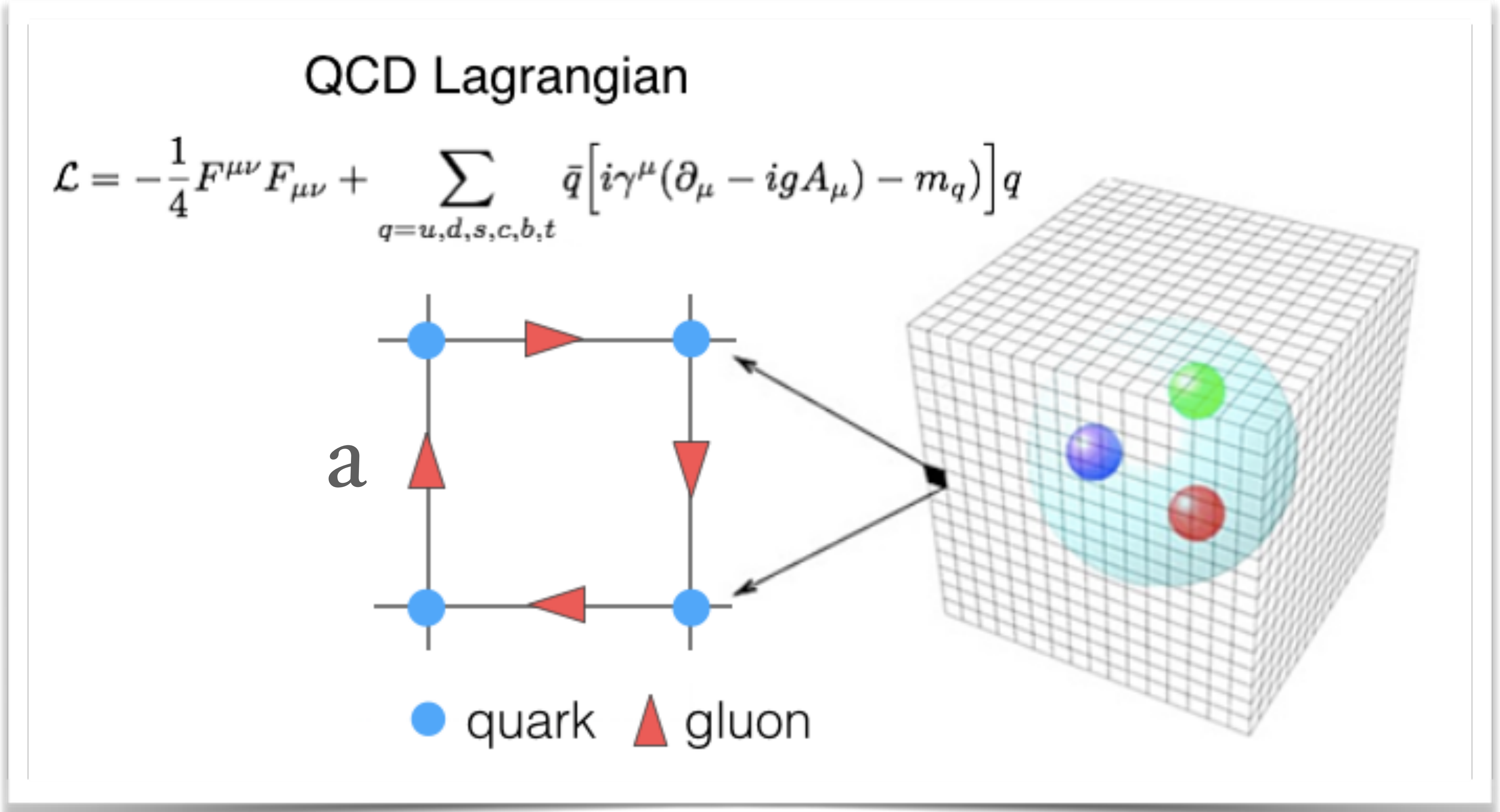
“physicist who only needs pencil and paper to do physics”

(and coffee)



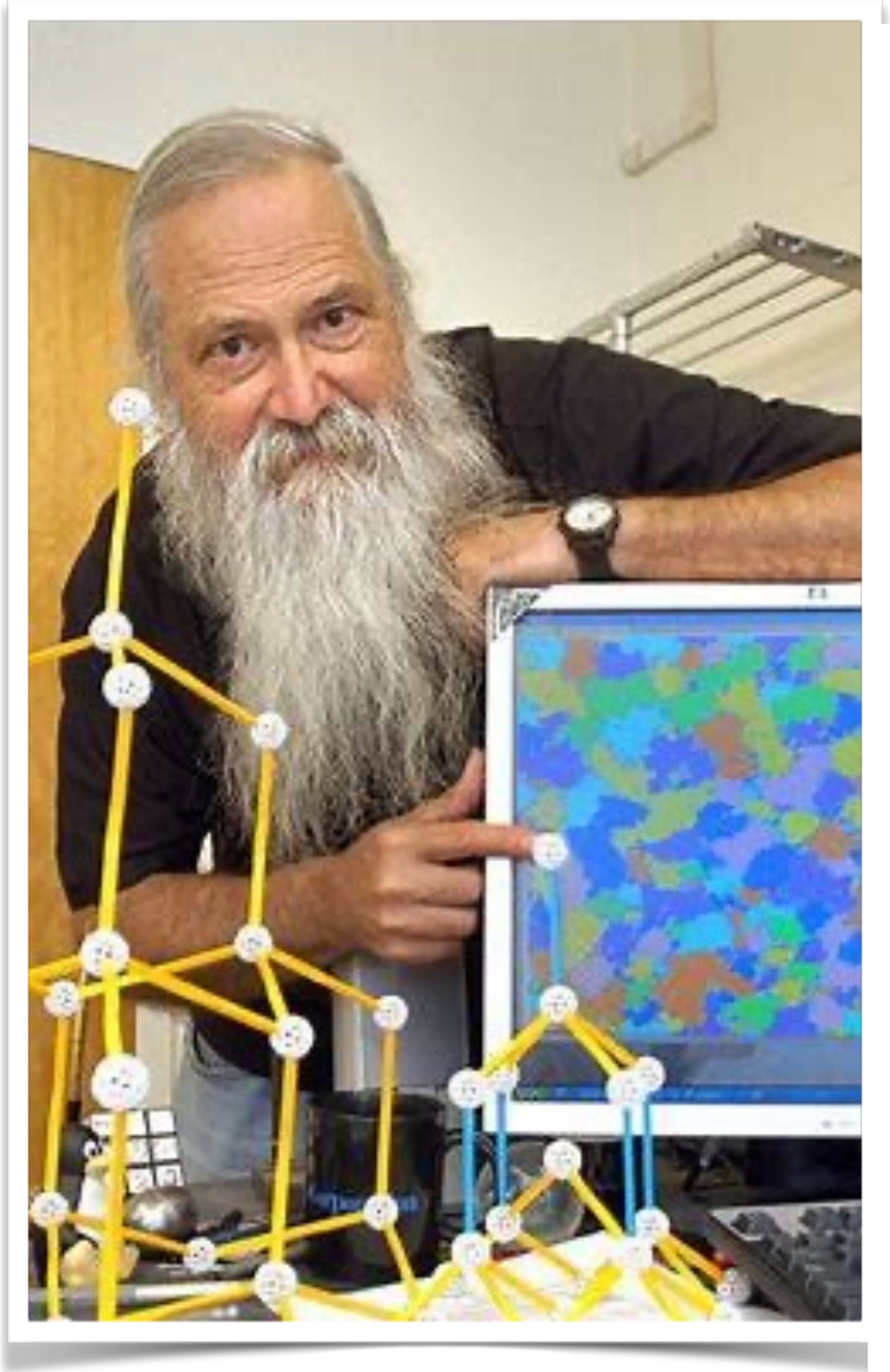
Kenneth G. Wilson

Lattice field theory

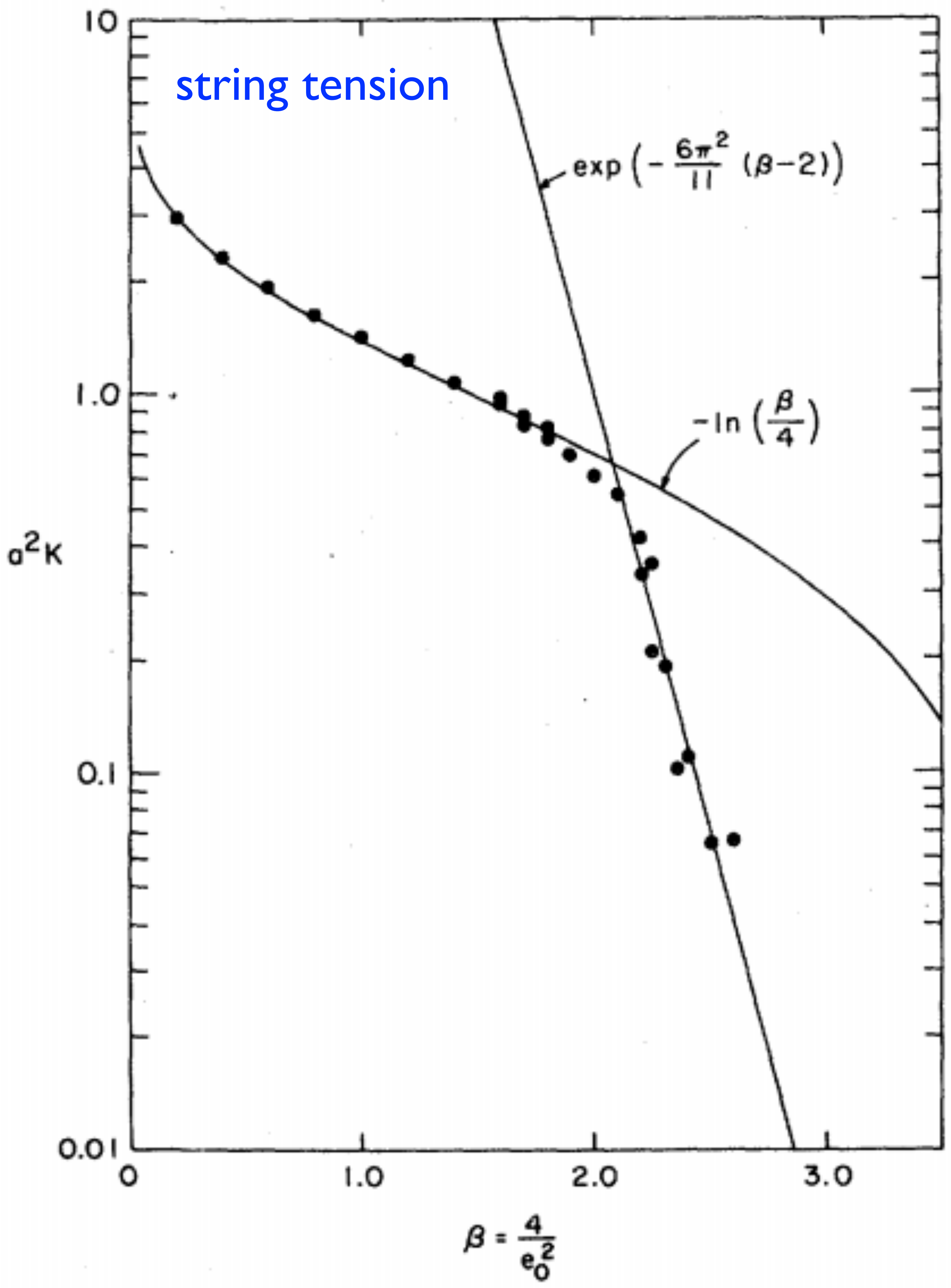


a→0 recovers QCD

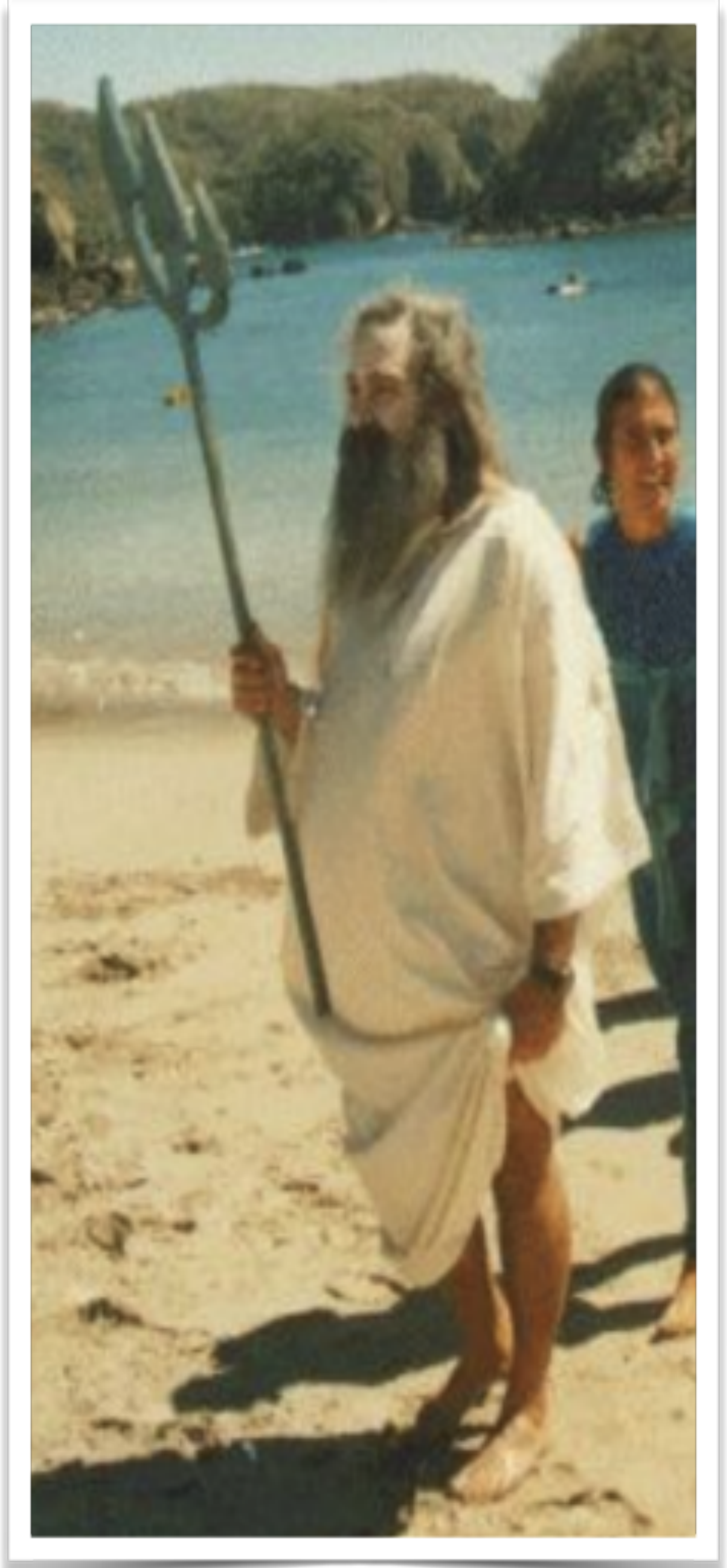
First numerical lattice simulations



Michale Creutz
@BNL



PRD 1980

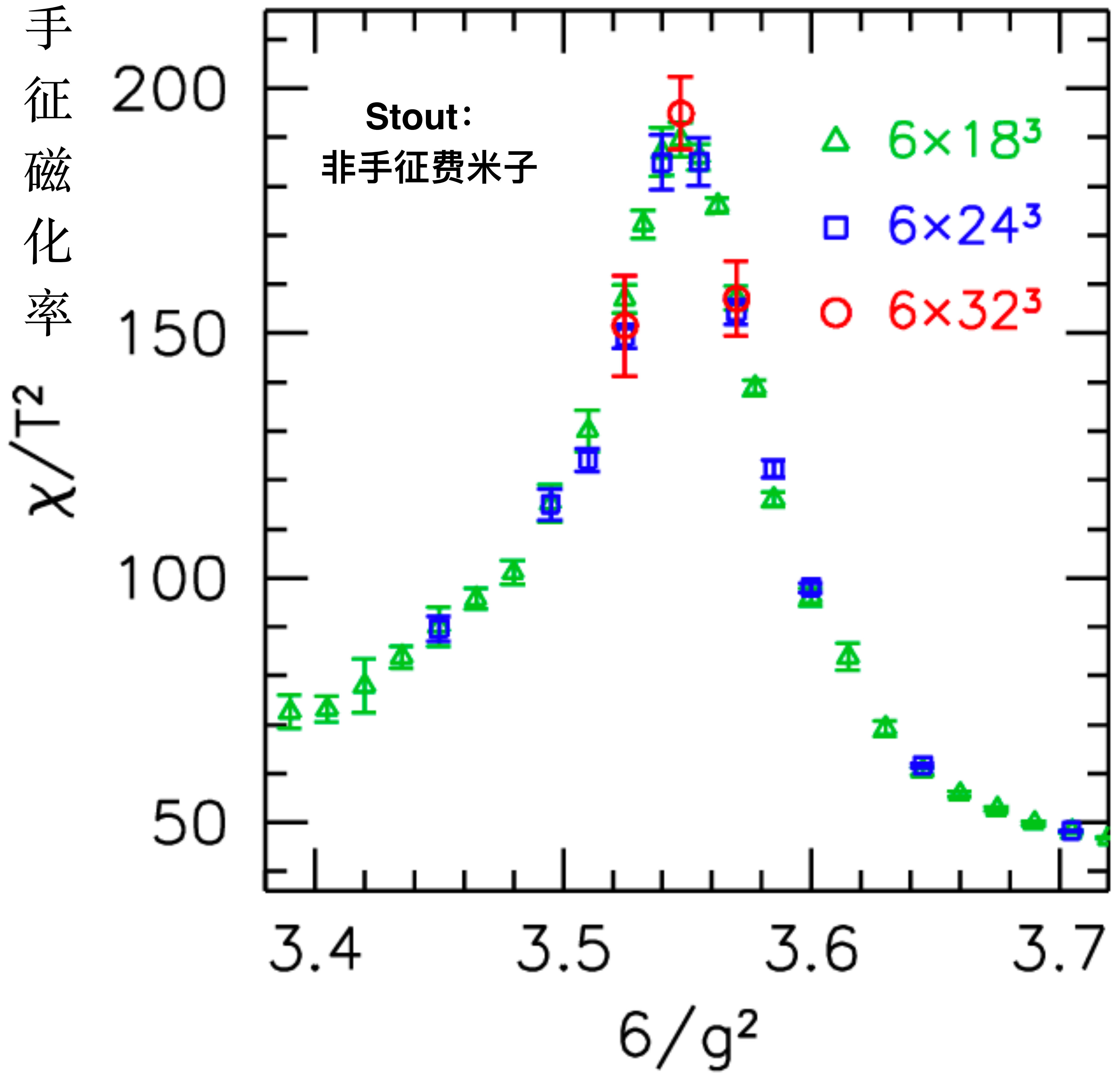


Michale Creutz
on the beach

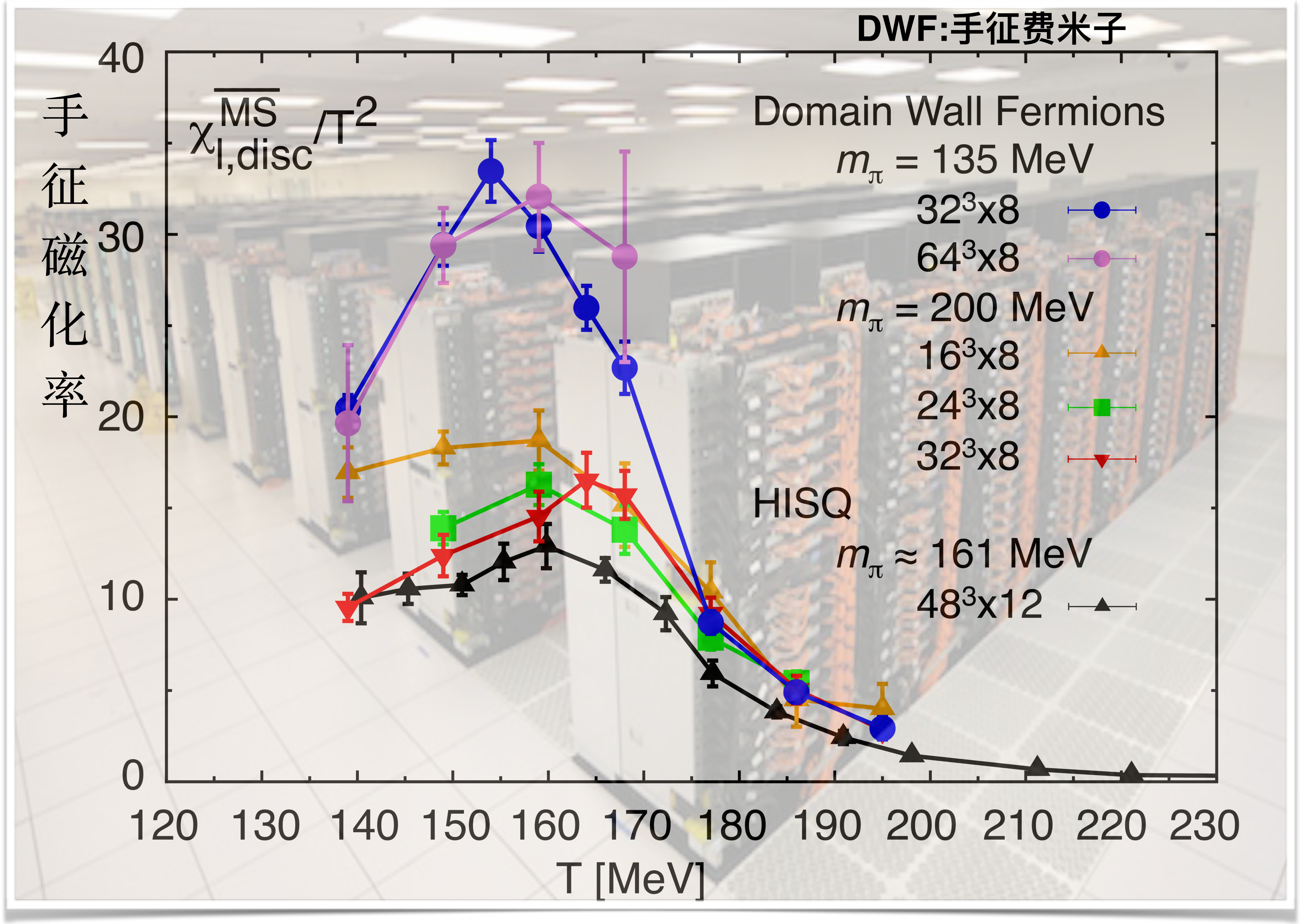
Spawned
golden age
in lattice QCD

Nature of QCD transition at the physical point

Crossover transition



Y. Aoki et al, Nature 443 (2006) 675-678

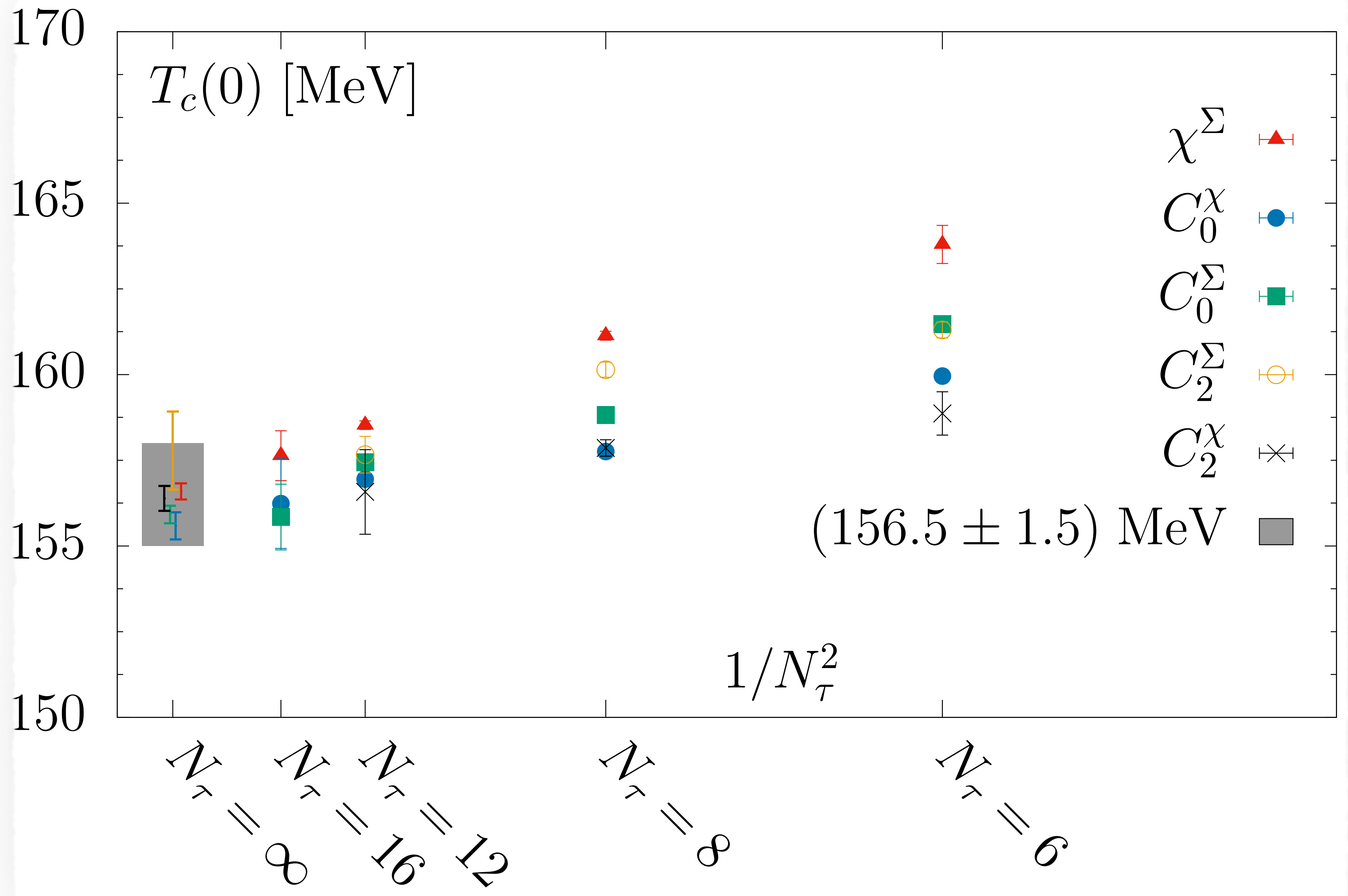


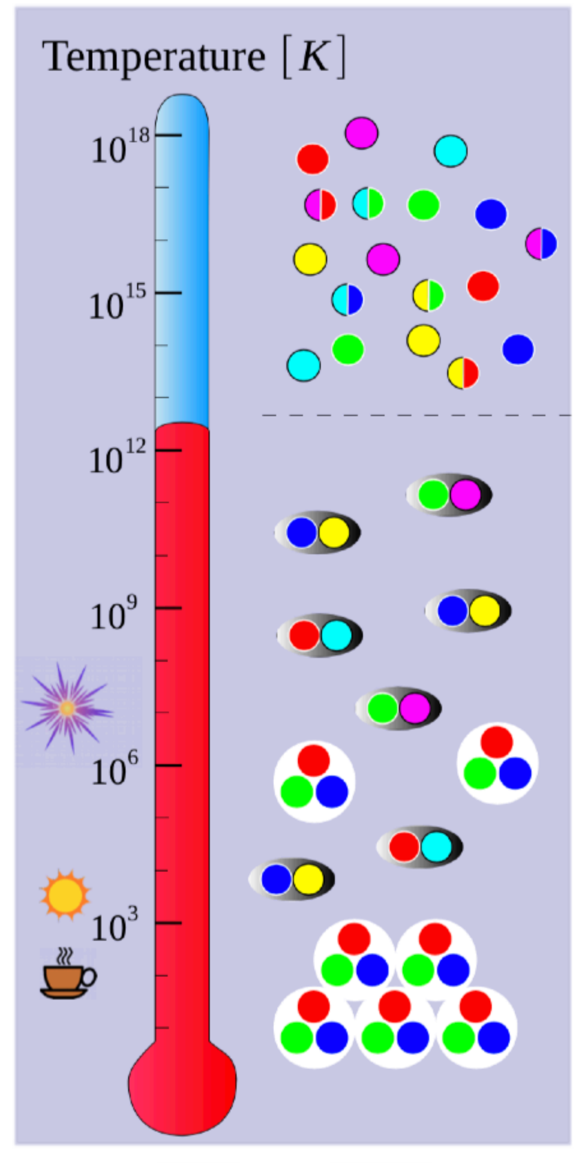
背景为美国Lawrence Livermore 国家实验室
的Sequoia超级计算机

T. Bhattacharya,…HTD,…et al.[HotQCD],
Phys. Rev. Lett.,113(2014)082001

Chiral crossover/phase transition temperature at physical point and $m_q \rightarrow 0$

Rigorous definition from O(4) universality class



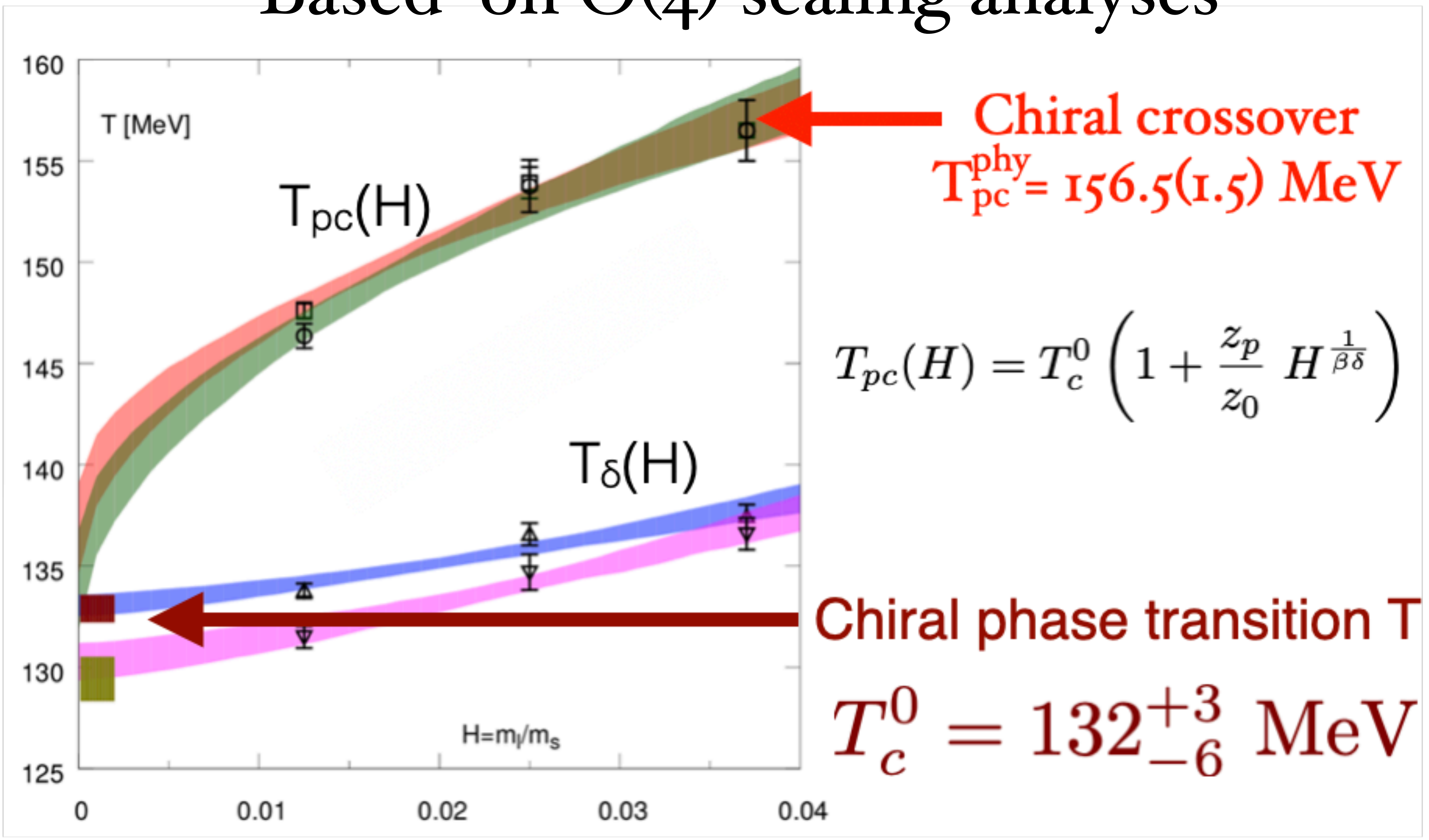


A. Bazavov, HTD, P. Hegde et al. [HotQCD], Phys. Lett. B795 (2019) 15

Chiral crossover transition
T=156.5(1.5) MeV

See also Wuppertal-Budapest, PRL125 (2020) 052001

Based on O(4) scaling analyses

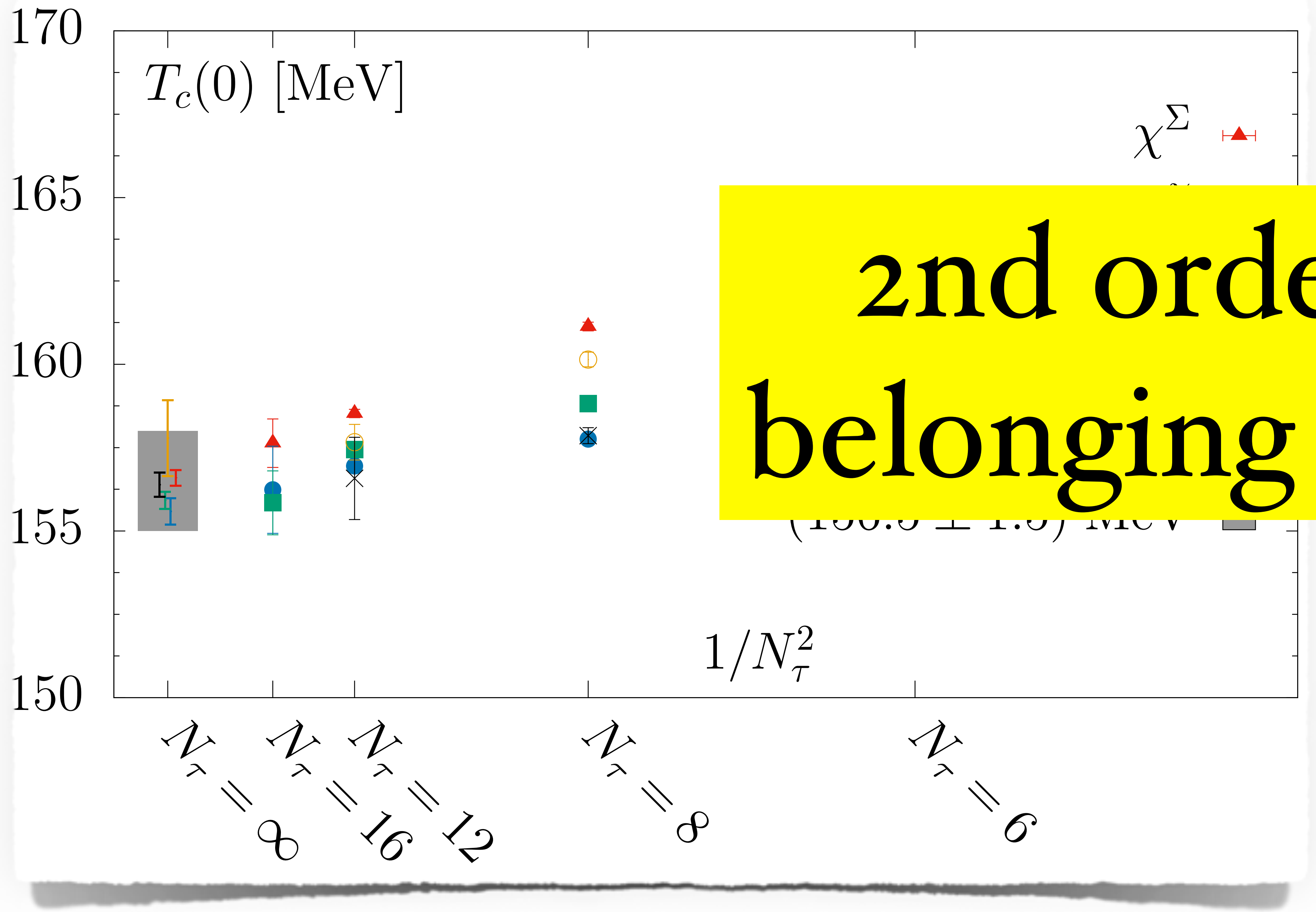


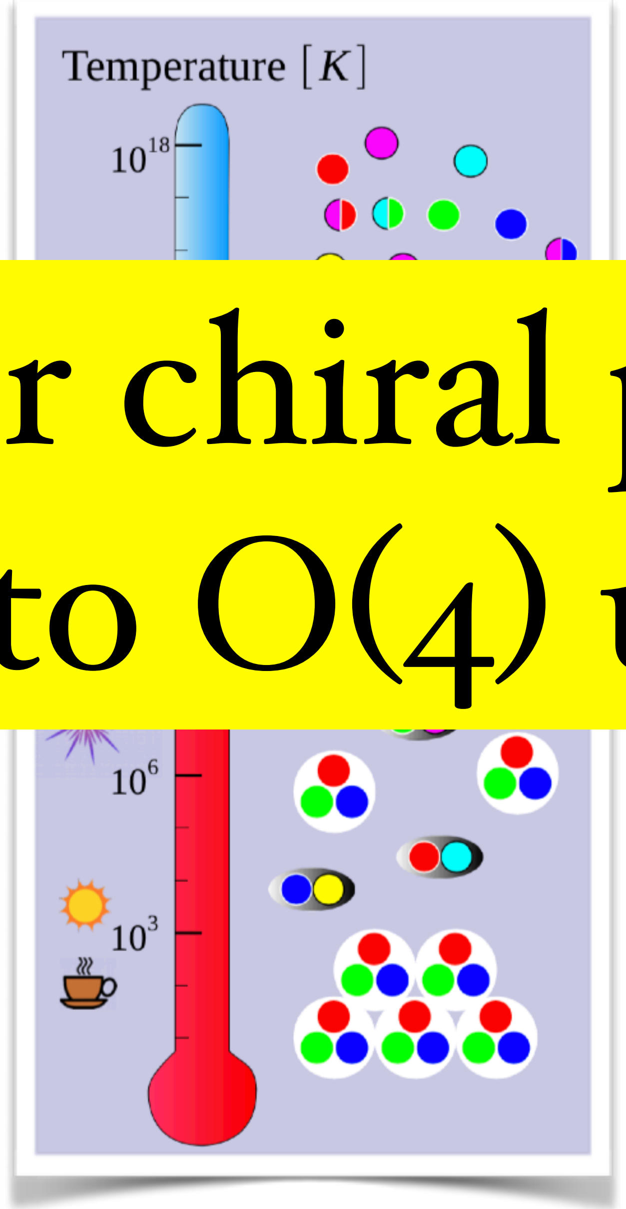
HTD, P. Hegde, O. Kaczmarek et al.[HotQCD], Phys. Rev. Lett. 123 (2019) 062002

Chiral phase transition T is about 25 MeV lower !

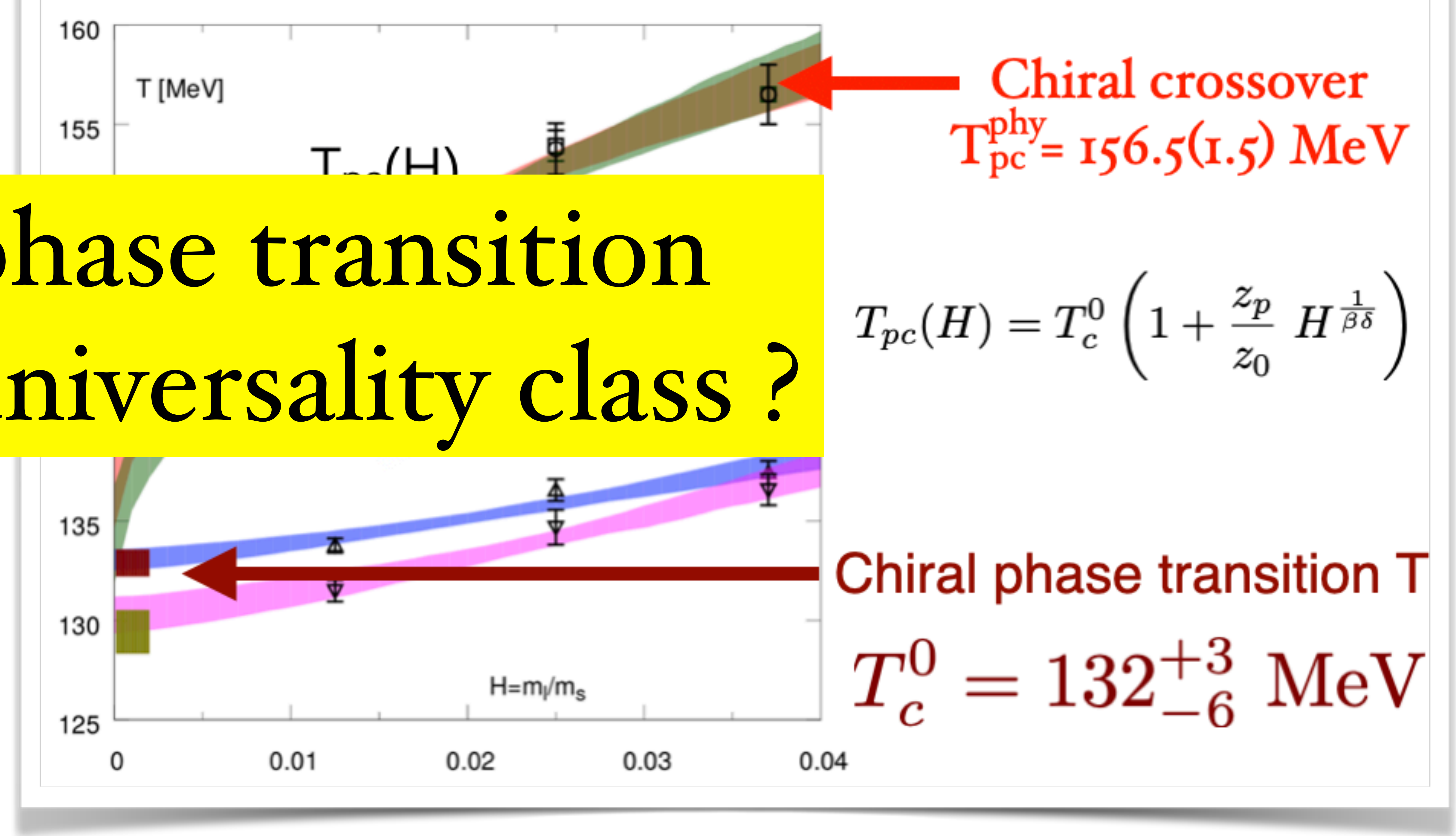
Chiral crossover/phase transition temperature at physical point and $m_q \to 0$

Rigorous definition from O(4) universality class





Based on O(4) scaling analyses



2nd order chiral phase transition belonging to O(4) universality class ?

A. Bazavov, HTD, P. Hegde et al. [HotQCD], Phys. Lett. B795 (2019) 15

Chiral crossover transition
T=156.5(1.5) MeV

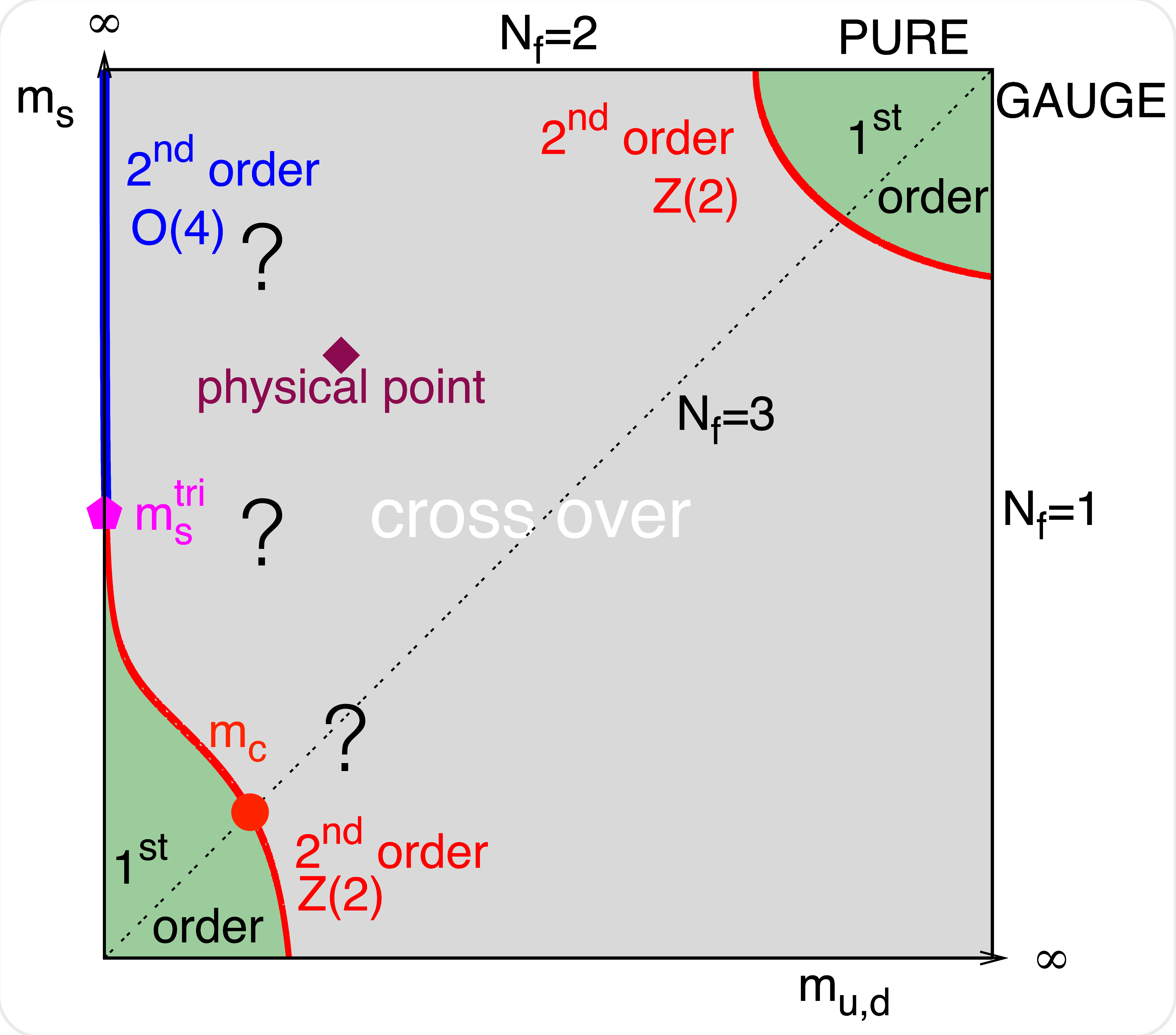
See also Wuppertal-Budapest, PRL125 (2020) 052001

HTD, P. Hegde, O. Kaczmarek et al.[HotQCD], Phys. Rev. Lett. 123 (2019) 062002

Chiral phase transition T is about 25 MeV lower !

Nature of chiral phase transition

Columbia plot:
QCD phase diagram in quark mass plane



- At physical point $T_{pc} \approx 156$ MeV HotQCD, PLB 795 (2019) 15; WB, PRL125 (2020) 052001
- Chiral phase transition T_c =132(+3)(-6)MeV

HotQCD, PRL 123 (2019) 062002

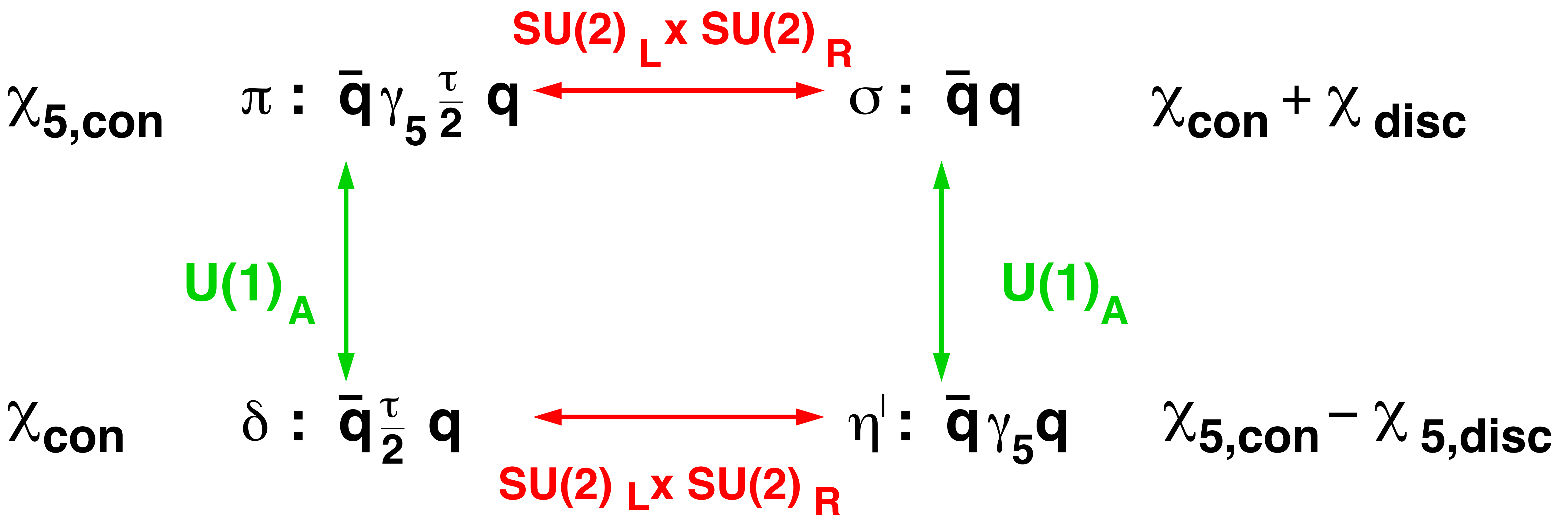
$U_A(1)$ symmetry:

Pisarski and Wilczek, PRD 29 (1984) 338
Butti, Pelissetto and Vicari, JHEP 08 (2003) 029
Pelissetto & Vicari, PRD 88 (2013) 105018
Grahl, PRD 90 (2014) 117904

- Broken, 2nd order (O(4)) phase transition
- Effectively restored, 1st or 2nd order $(U(2)_L \otimes U(2)_R/U(2)_V)$

Signatures of symmetry restorations

- Susceptibilities defined as integrated two point correlation functions of the local operators, e.g. $\chi_\pi = \int d^4x \langle \pi^i(x)\pi^i(0)\rangle$ with $\pi^i(x) = i\bar{\psi}_l(x)\gamma_5\tau^i\psi_l(x)$



$$\chi_{\rm disc} = \frac{T}{V}\int d^4x \left\langle \left[\bar{\psi}(x)\psi(x) - \langle\bar{\psi}(x)\psi(x)\rangle\right]^2 \right\rangle$$

Restoration of $SU(2)_L x SU(2)_R$:

$\chi_\pi - \chi_\sigma = 0$

$\chi_\delta - \chi_\eta = 0$

$\Rightarrow \chi_\pi - \chi_\delta = \chi_{disc} = \chi_{5,disc}$

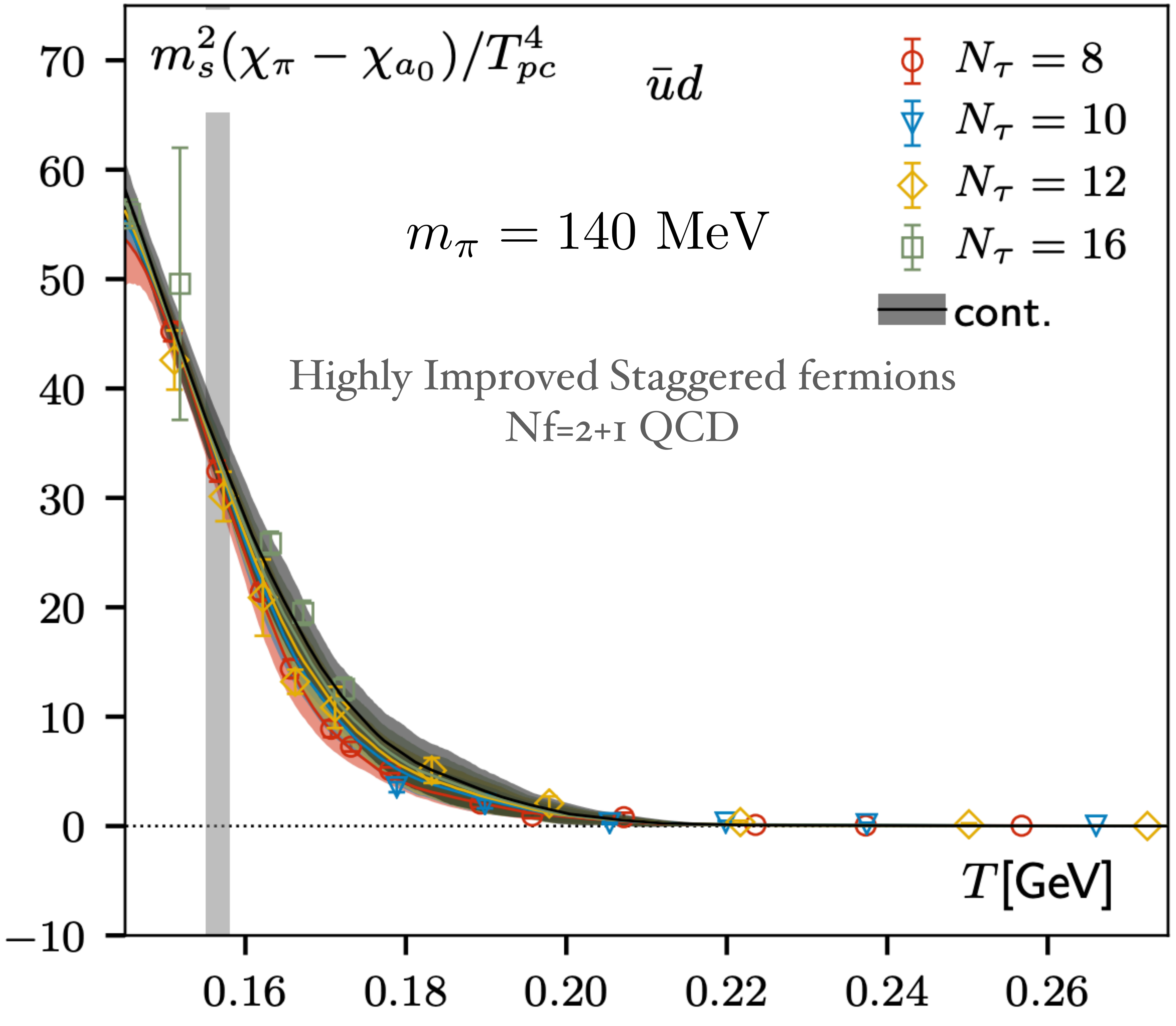
Effective restoration of $U(1)_A$:

$\chi_\pi - \chi_\delta = 0$

$\chi_\sigma - \chi_\eta = 0$

$\Rightarrow \chi_\pi - \chi_\delta = \chi_{disc} = \chi_{5,disc} = 0$

Status of lattice studies on axial anomaly



HotQCD, Phys.Rev.D 100 (2019) 094510

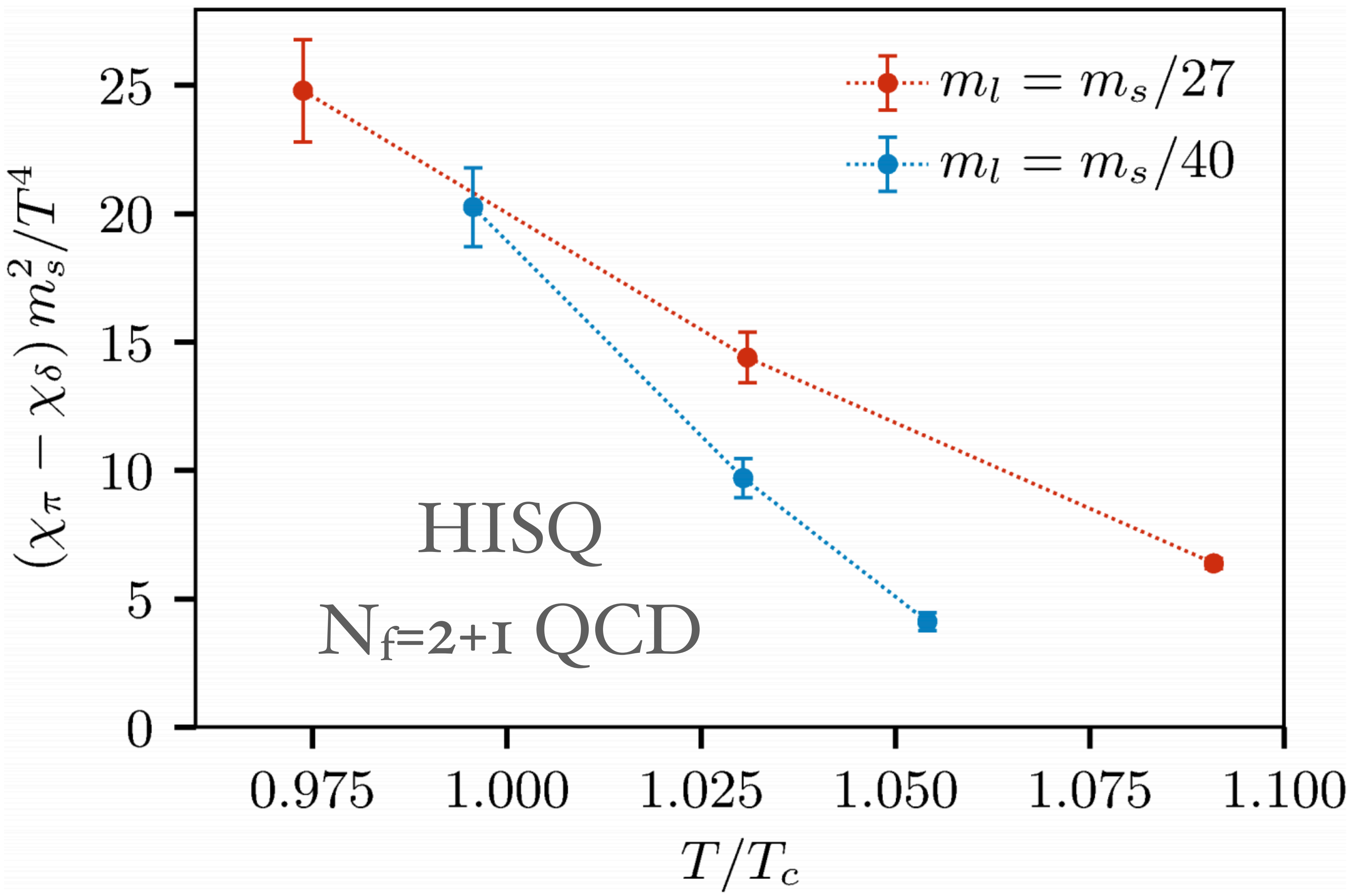
- ☑ At physical pion mass at T ≤ T_{pc} axial anomaly remains manifested in χ_π - χ_δ

See similar conclusions obtained using chiral fermions:
HotQCD, PRL 113(2014)082001, PRD 89 (2014)054514
JLQCD, arXiv: 2011.01499, ...

- ☐ How about the case in the chiral limit ?

Axial anomaly towards chiral limit

Nt=8, lattice spacing a≈0.15 fm

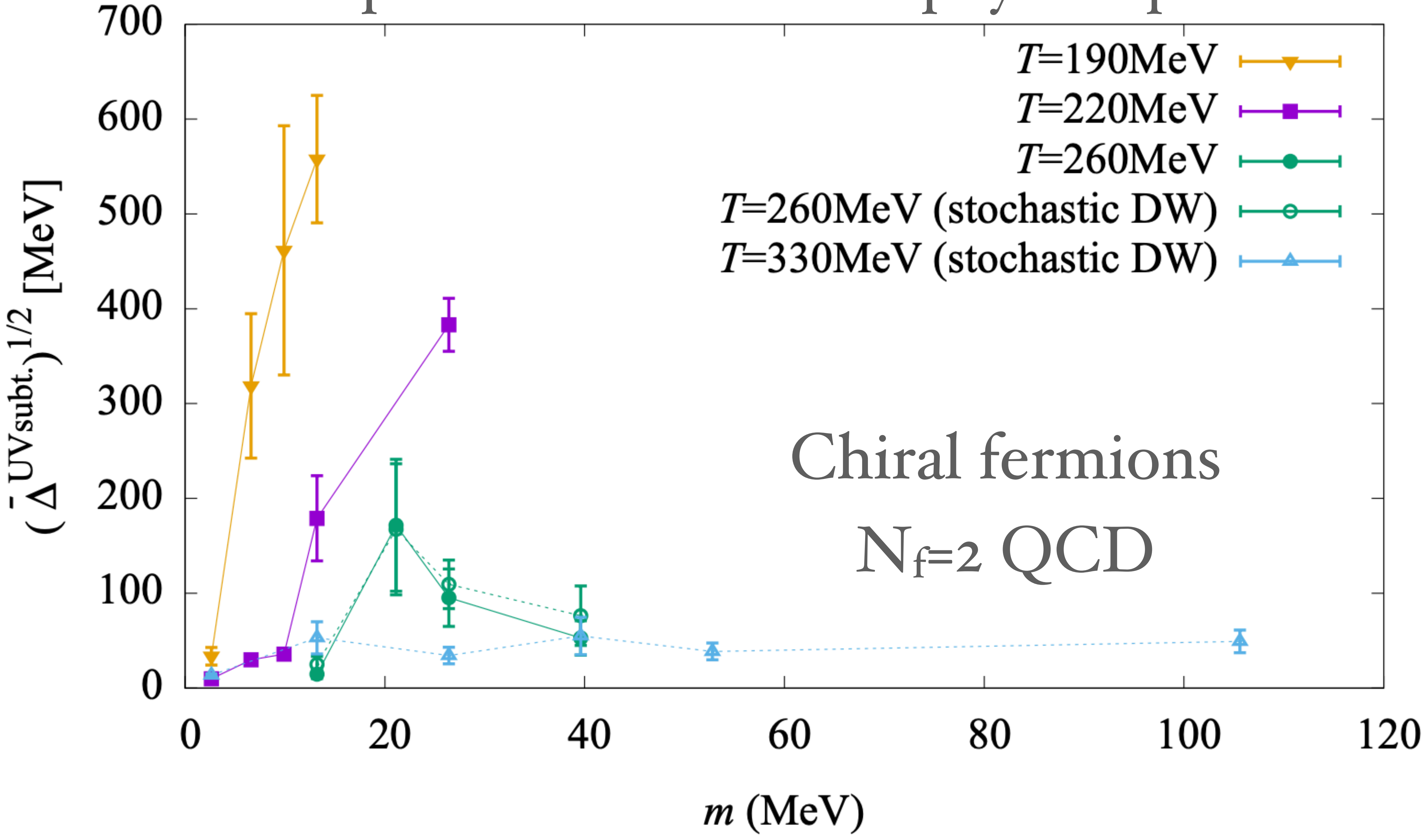


Sharma, Lattice 2018 Review talk, 1901.07190

Remains manifested at m_π=110 MeV and T<1.1T_c

Similar conclusions from
Dick et al., PRD 91(2015)094504, Ohno et al., PoS Lattice 2012(2012)095, Mazur et al., 1811.08222,...

Fixed scale approach, lattice spacing a=0.074 fm
One quark mass below the physical point



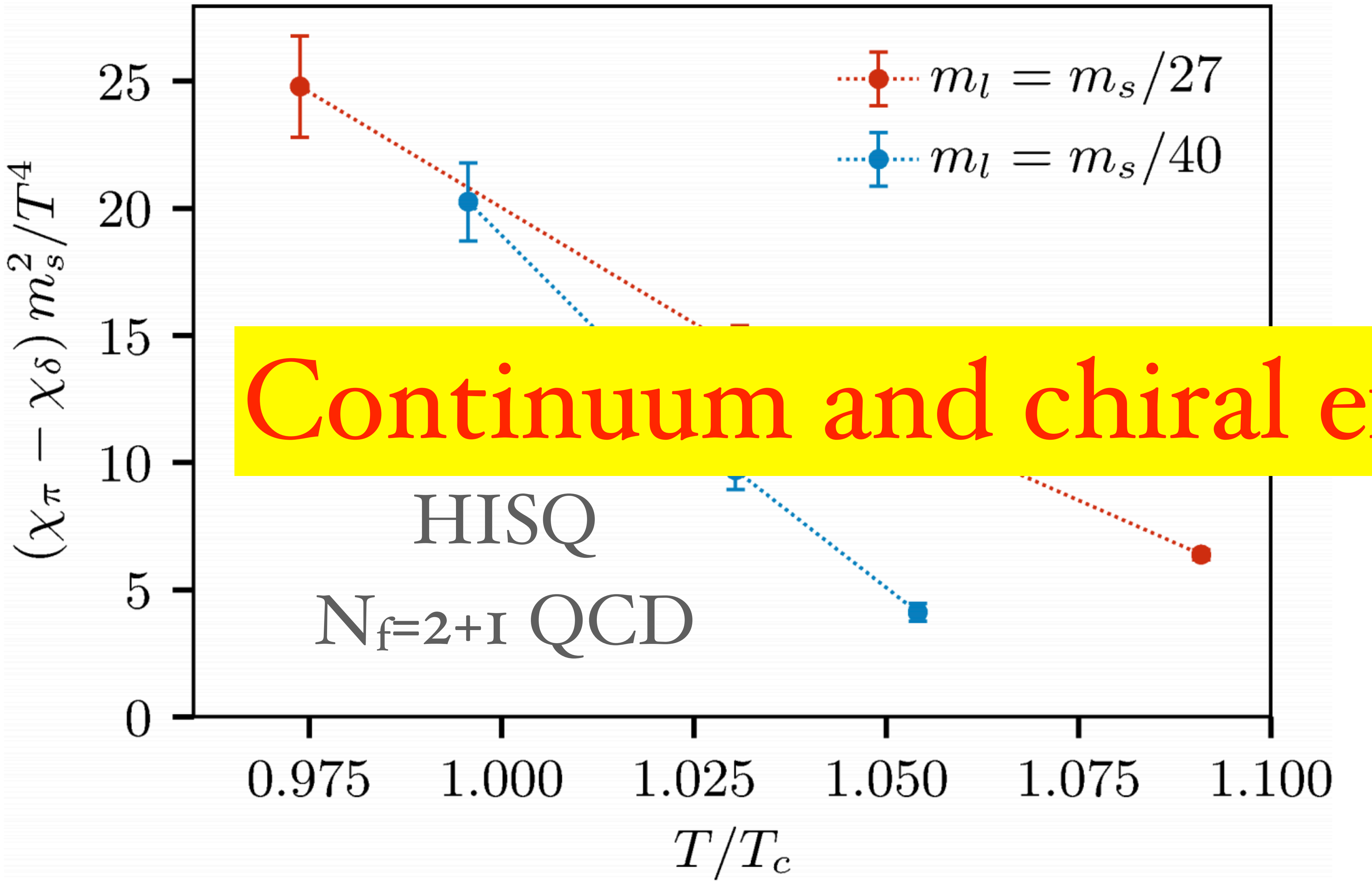
JLQCD, arXiv: 2011.01499

Seems to disappear at T ≳ 220 MeV

Similar conclusions from
Chiu et al.,PoS Lattice 2013 (2014)165,
Tomiya et al.,[JLQCD] PRD 96(2017)079902,
Brandt et al., JHEP 12 (2016) 158,..

Axial anomaly towards chiral limit

Nt=8, lattice spacing a≈0.15 fm



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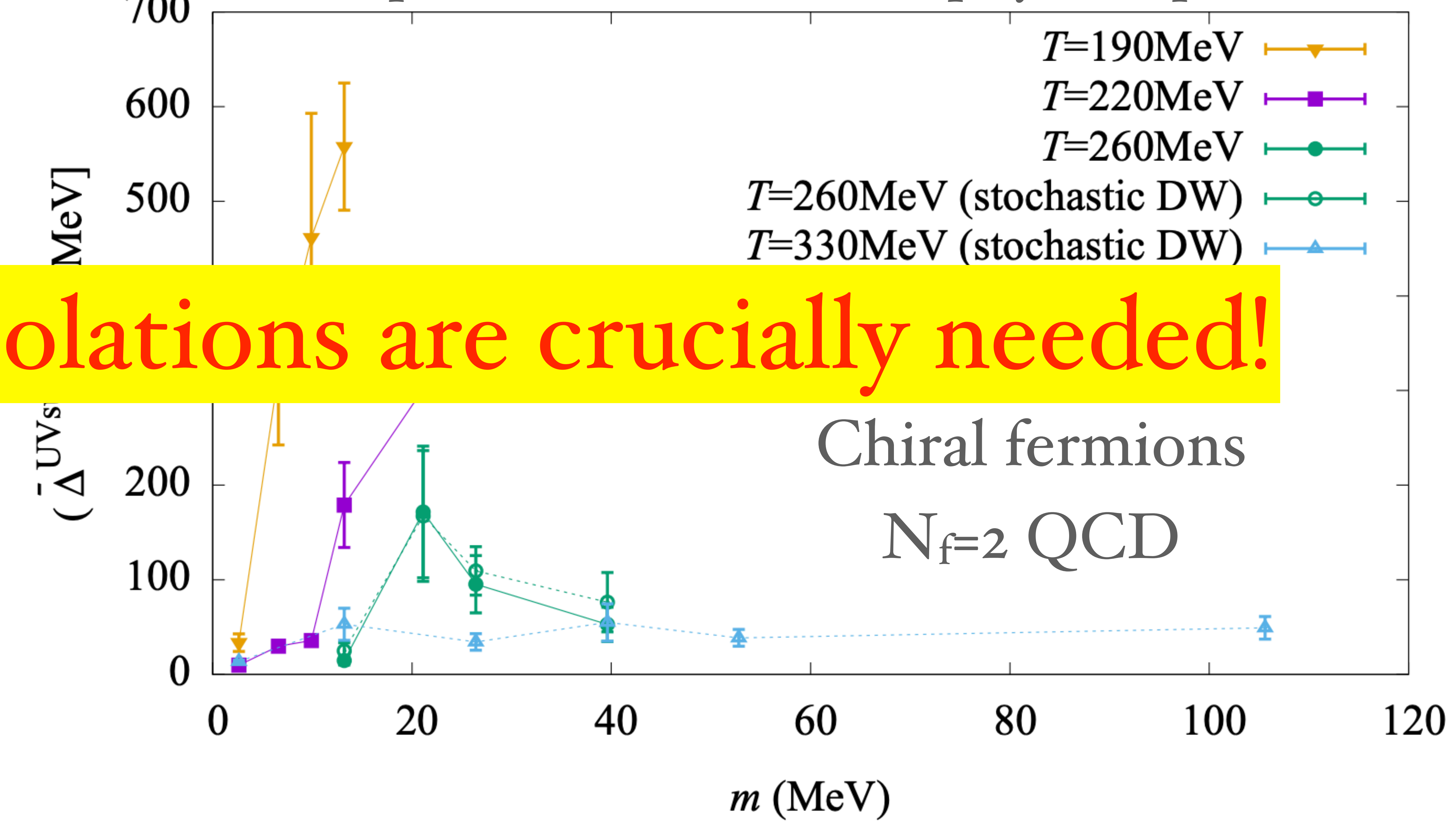

Continuum and chiral extrapolations are crucially needed!

JLQCD, arXiv: 2011.01499

Remains manifested at m_π=110 MeV and T<1.1T$_c$

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Dick et al., PRD 91(2015)094504, Ohno et al., PoS Lattice 2012(2012)095,
Mazur et al., 1811.08222,...

Similar conclusions from
Chiu et al.,PoS Lattice 2013 (2014)165,
Tomiya et al.,[JLQCD] PRD 96(2017)079902,
Brandt et al., JHEP 12 (2016) 158,..

Signature of restorations in Dirac Eigenvalue Spectrum

$$\langle\bar{\psi}\psi\rangle = \int_0^\infty \frac{4m_l\,\rho}{\lambda^2+m_l^2}\,\mathrm{d}\lambda\,, \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda\,\frac{8m_l^2\,\rho}{\left(\lambda^2+m_l^2\right)^2}$$

- Restoration of $SU(2)_L x SU(2)_R$ symmetry
  - $\rho(0)=0$ as from Banks-Casher formula: $\lim_{m_l\to 0}\langle\bar{\psi}\psi\rangle = \pi\rho(0)$ Banks and Casher, NPB 169 (1980) 103
  - Partition function is an even function in quark mass due to the Z(2) subgroup
- Effective restoration of $U(1)_A$ symmetry
  - a sizable gap from zero, i.e. $\rho(\lambda<\lambda_c)=0$ Cohen, arXiv:nucl-th/9801061

⚠ if $\rho(\lambda)$ is analytic in m^2, NOT be manifested in differences of up to 6 point correlation functions Aoki, Fukaya and Taniguchi, PRD86 (2012) 114512

Possible behavior of $\rho(\lambda)$ making SU(2)xSU(2) restored but NOT $U(1)_A$

$$\rho(\lambda,m) = c_0 + c_1\lambda + c_2 m^2 \delta(\lambda) + c_3 m + c_4 m^2 + O(\lambda,m)$$

$$\langle\bar\psi\psi\rangle = 2c_0\pi - 4c_1 m\ln(m) + 2c_2 m + 2\pi c_3 + 2\pi c_4 m^2$$

$$\chi_\pi - \chi_\delta = 2c_0\pi/m + 4c_1 + 4c_2 + 2\pi c_3 + 2\pi c_4 m$$

Ansatz	$\langle\bar\psi\psi\rangle$	χ_π	χ_δ	$\chi_\pi - \chi_\delta$	χ_{disc}
c	$2c\pi$	$2c\pi/m$	0	$2c\pi/m$	0
λ	$-4m\ln(m)$	$-4\ln(m)$	$-4\ln(m)$	4	0
$m^2\delta(\lambda)$	$2m$	2	-2	4	4
m	$2\pi m$	2π	0	2π	2π
m^2	$2\pi m^2$	$2\pi m$	0	$2\pi m$	$2\pi m$

HotQCD, PRD86(2012)094503

c_0 & c_1 terms: break both symmetries Smilga & Stern, PLB 93'

c_2: near zero mode contribution Gross, Yaffe & Pisarski, RMP 81'

c_3: another $U(1)_A$ breaking term

c_4: Not manifested in 2-pt correlators Aoki, Fukaya & Taniguchi, PRD12'

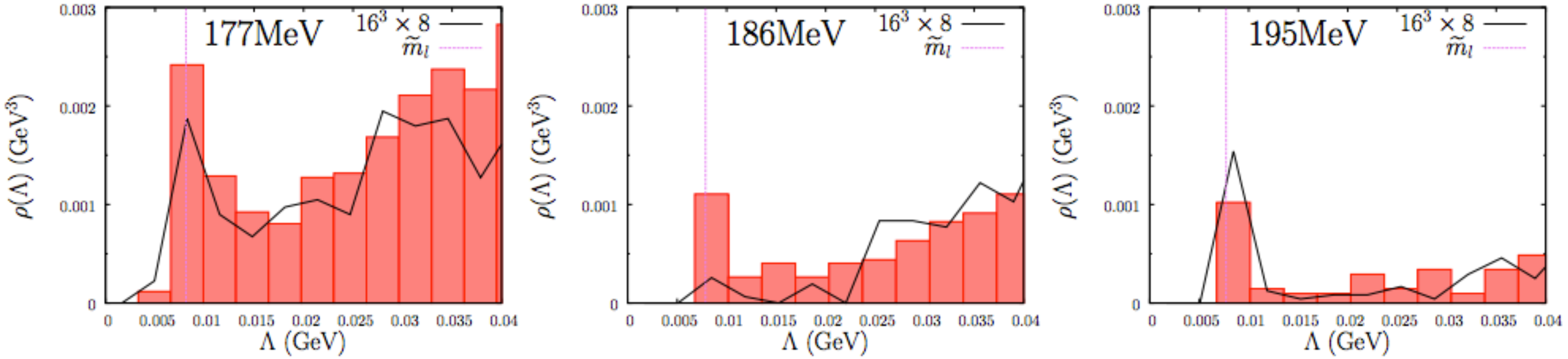
- LQCD: At high T for physical *m*, the T dependence of χ_t follows dilute instanton gas approximation prediction See recent review: Lombardo & Trunin, IJMPA 35(2020)2030010

Due to $\rho(\lambda,m) \propto m^2\delta(\lambda)$? What happens for $m \to 0$?

Infrared enhancement in ρ

LQCD simulations of N_f=2+1 QCD using Domain Wall fermions, m_π=200 MeV

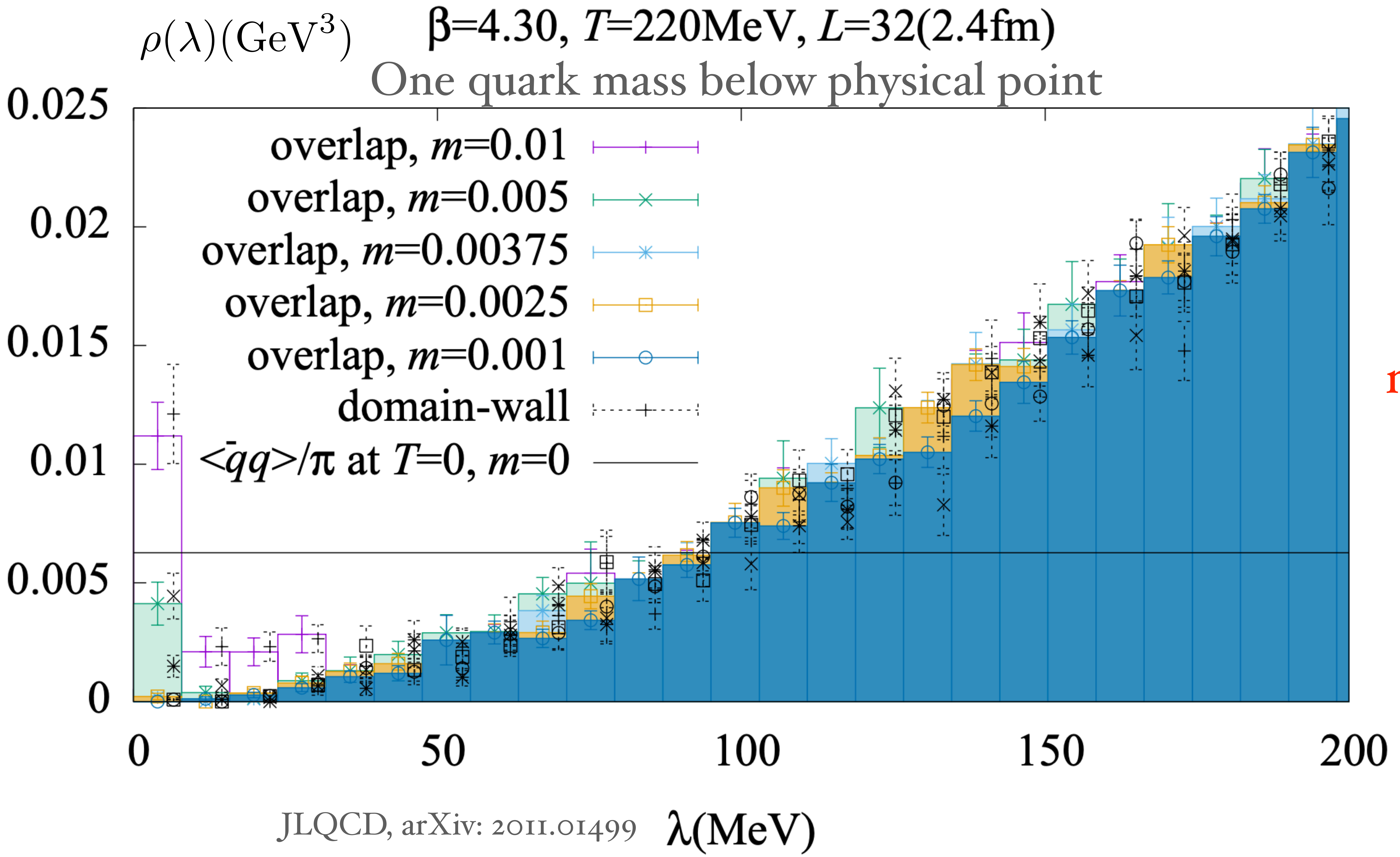
black lines: results from 16^3 lattices, red histograms: results from 32^3 lattices



HotQCD, PRD 89 (2014)054514

The m_l^2 dependence is not demonstrated as $m_l \rightarrow 0$

No infrared enhancement in ρ



JLQCD, arXiv: 2011.01499

No clear gap

At m<0.01 and λ>0, m dependence can be hardly seen

Continuum extrapolation is important

So far…

- Mass dependence of rho ?
- Continuum and chiral extrapolations ?

Novel relation: Light quark mass derivative of ρ and C_n

$$\rho(\lambda, m_l) = \frac{T}{VZ[\mathcal{U}]}\int \mathcal{D}[\mathcal{U}] e^{-S_G[\mathcal{U}]} \det\left[\not{D}[\mathcal{U}] + m_s\right] \left(\det\left[\not{D}[\mathcal{U}] + m_l\right]\right)^2 \rho_U(\lambda)$$

Partition function $\quad Z[\mathcal{U}] = \int \mathcal{D}[\mathcal{U}] e^{-S_G[\mathcal{U}]} \det\left[\not{D}[\mathcal{U}] + m_s\right] \left(\det\left[\not{D}[\mathcal{U}] + m_l\right]\right)^2$

Eigenvalue spectrum per ensemble $\quad \rho_U(\lambda) = \sum_j \delta(\lambda - \lambda_j)$

Quark mass dependence of ρ is enclosed in

$$\det\left[\not{D}[\mathcal{U}] + m_l\right] = \prod_j (+\mathrm{i}\,\lambda_j + m_l)(-\mathrm{i}\,\lambda_j + m_l) = \exp\left(\int_0^\infty \mathrm{d}\lambda\, \rho_U(\lambda) \ln\left[\lambda^2 + m_l^2\right]\right)$$

$$\Rightarrow \quad \frac{V}{T}\frac{\partial \rho}{\partial m_l} = \int_0^\infty \mathrm{d}\lambda_2\, \frac{4 m_l C_2}{\lambda_2^2 + m_l^2}$$

$$C_2(\lambda, \lambda_2) = \left\langle \rho_U(\lambda) \rho_U(\lambda_2) \right\rangle - \left\langle \rho_U(\lambda) \right\rangle \left\langle \rho_U(\lambda_2) \right\rangle$$

Relation between ρ derivatives and C_{n+1}

$$\frac{V}{T}\frac{\partial^2\rho}{\partial m_l^2} = \int_0^\infty d\lambda_2 \, \frac{4(\lambda_2^2 - m_l^2)\, C_2}{\left(\lambda_2^2 + m_l^2\right)^2} + \int_0^\infty d\lambda_2 \, d\lambda_3 \, \frac{(4m_l)^2\, C_3}{(\lambda_2^2 + m_l^2)(\lambda_3^2 + m_l^2)}$$

••• •••

••• •••

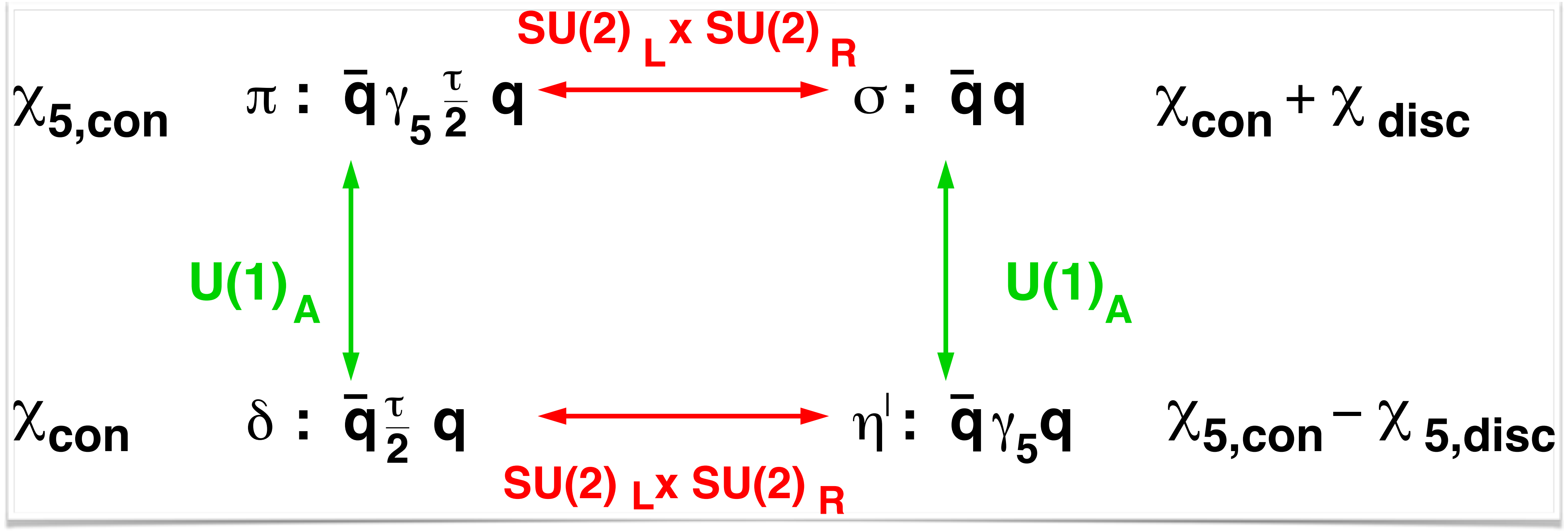
$$C_n(\lambda_1, \cdots, \lambda_n; m_l) = \left\langle \prod_{i=1}^{n} \left[\rho_U(\lambda_i) - \langle \rho_U(\lambda_i) \rangle\right] \right\rangle$$

Signatures of symmetry restorations

- Chiral symmetry restoration: $\chi_\pi - \chi_\delta = \chi_{\rm disc}$

$$\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \, \frac{8 m_l^2 \rho}{(\lambda^2 + m_l^2)^2} \, ,$$

$$\chi_{\rm disc} = \int_0^\infty d\lambda \, \frac{4 m_l \, \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$



Toublan and Verbaarschot, NPB603 (2001) 343

HotQCD, PRD90 (2014) 094503

Kanazawa & Yamamoto, JHEP 01(2016)141

- If eigenvalues are uncorrelated $C_n^{\rm Po}(\lambda_1, \cdots, \lambda_n) = \delta(\lambda_1 - \lambda_2) \cdots \delta(\lambda_n - \lambda_{n-1}) \langle \rho_U(\lambda_1) \rangle + \mathcal{O}(1/N)$

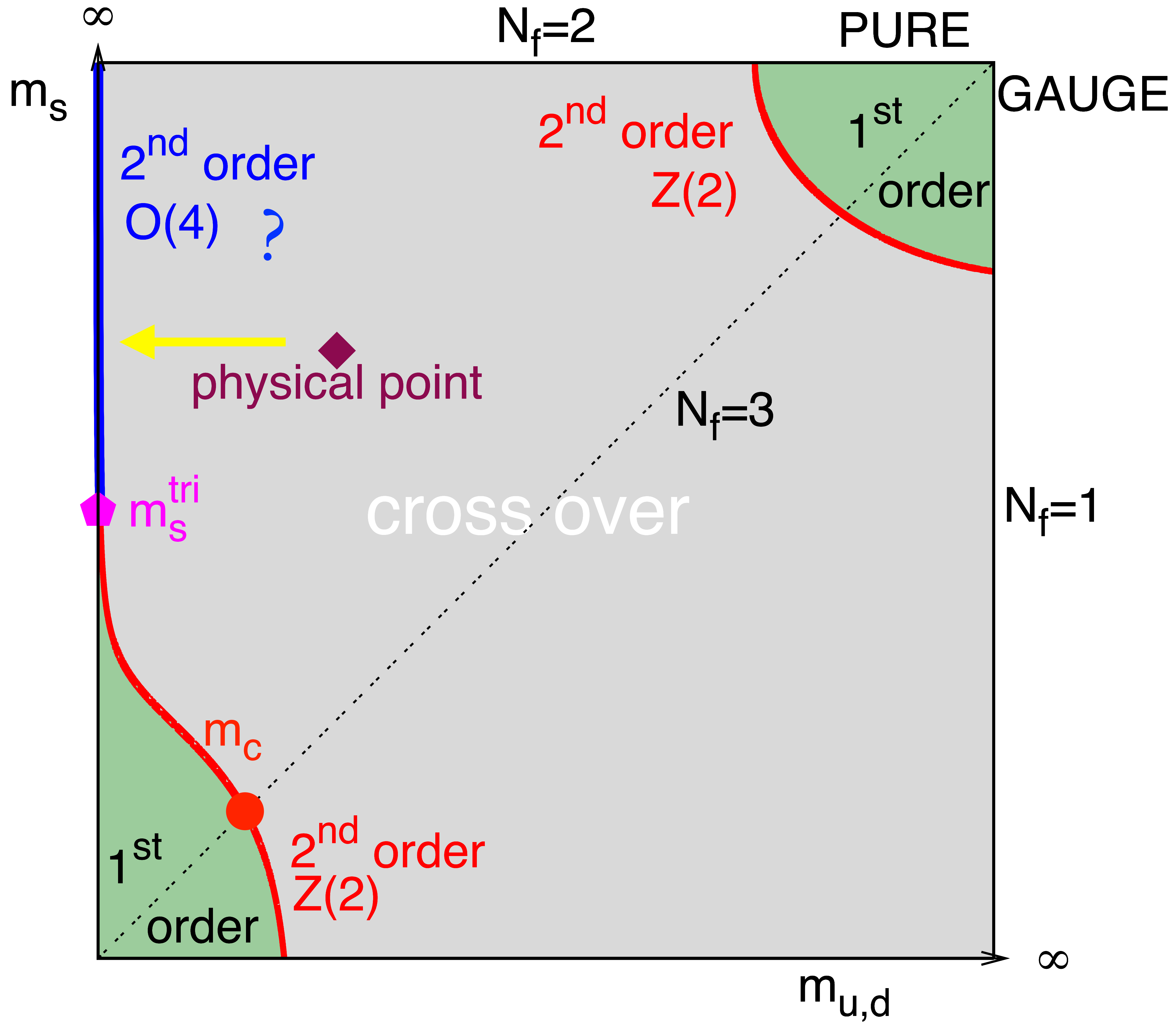
$$\left(\frac{\partial \rho}{\partial m_l} \right)^{\rm Po} = \frac{4 m_l \rho}{\lambda^2 + m_l^2} - \frac{V \rho}{TN} \langle \bar{\psi} \psi \rangle \Rightarrow \chi_{\rm disc}^{\rm Po} = 2(\chi_\pi - \chi_\delta)$$

Non-Poisson correlation among eigenvalues:
needed for chiral symmetry restoration if $\chi_\pi - \chi_\delta \neq 0$

Kanazawa & Yamamoto, JHEP 01(2016)141

Lattice setup

local GPU computers @CCNU



HTD, S.-T. Li, A. Tomiya, S. Mukherjee, X.-D. Wang, Y. Zhang
arXiv: 2011.04870

- At a single T=205 MeV
- HISQ/tree action
- N_f=2+1:
  - ☑ Nt=8,12,16 (a=0.12,0.08,0.06 fm)
  - ☑ m_s^{phy}/m_l = 20, 27, 40, 80, 160
    $m_\pi \approx$160,140,110, 80, 55 MeV
  - ☑ $9 \geq N_s/N_t \geq 4$

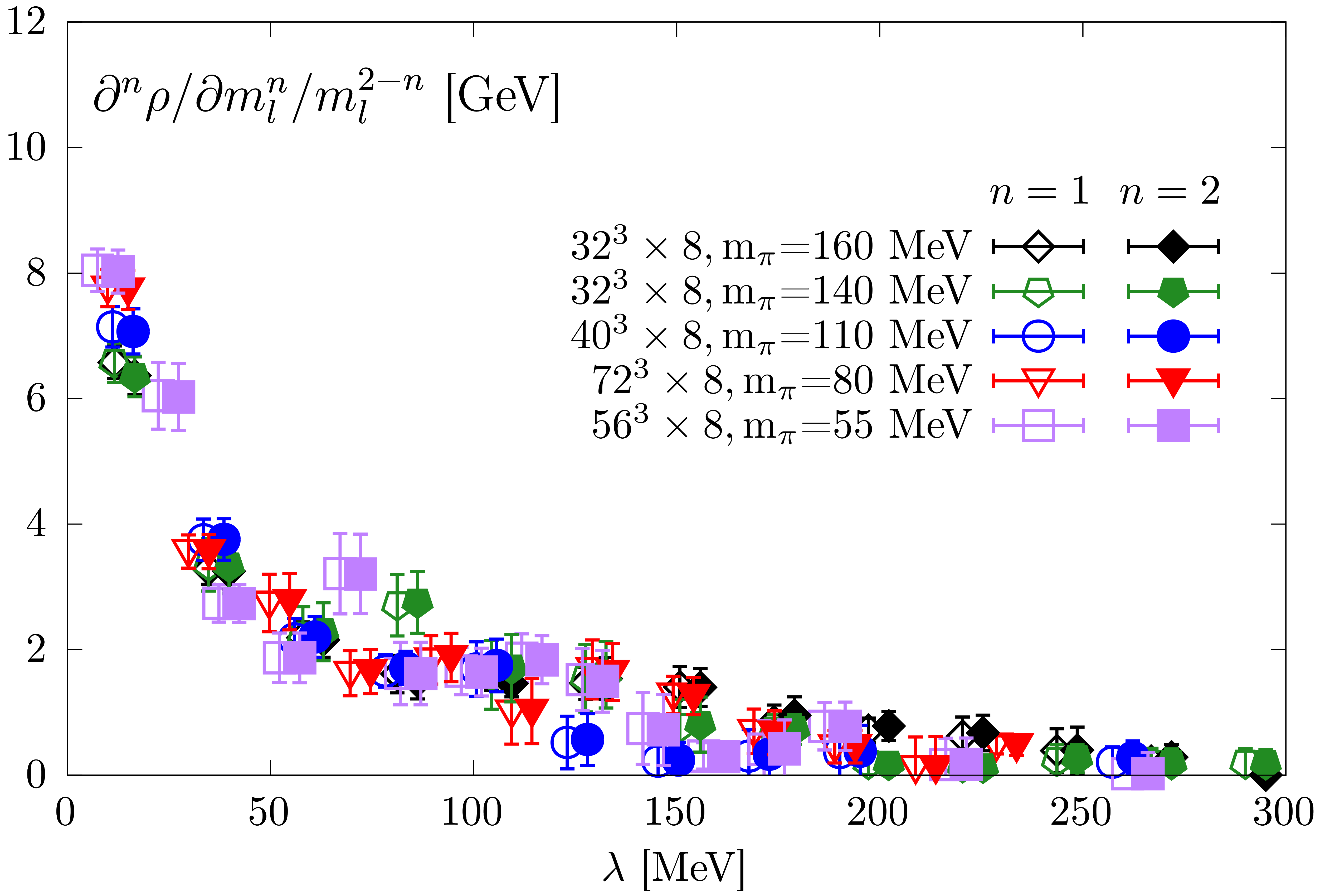
Complete eigenvalue spectrum ρ and C_2 at T=205 MeV

HTD, S.-T. Li, S. Mukherjee, A. Tomiya, X.-D. Wang, Y. Zhang, arXiv:2010.14836

via Chebyshev Polynomial filtering technique

Giusti and Luscher, JHEP03(2009)013, Patella PRD86(2012)025006, Cossu et al., PTEP 2016(2016)093B06
Itou & Tomiya, arXiv:1411.1155, Fodor et al., arXiv:1605.08091, de Forcrand & Jäger, arXiv: 1710.07305, HTD et al., arXiv:2001.05217,2008.00493

1st & 2nd mass derivative of ρ on N_τ=8 lattices



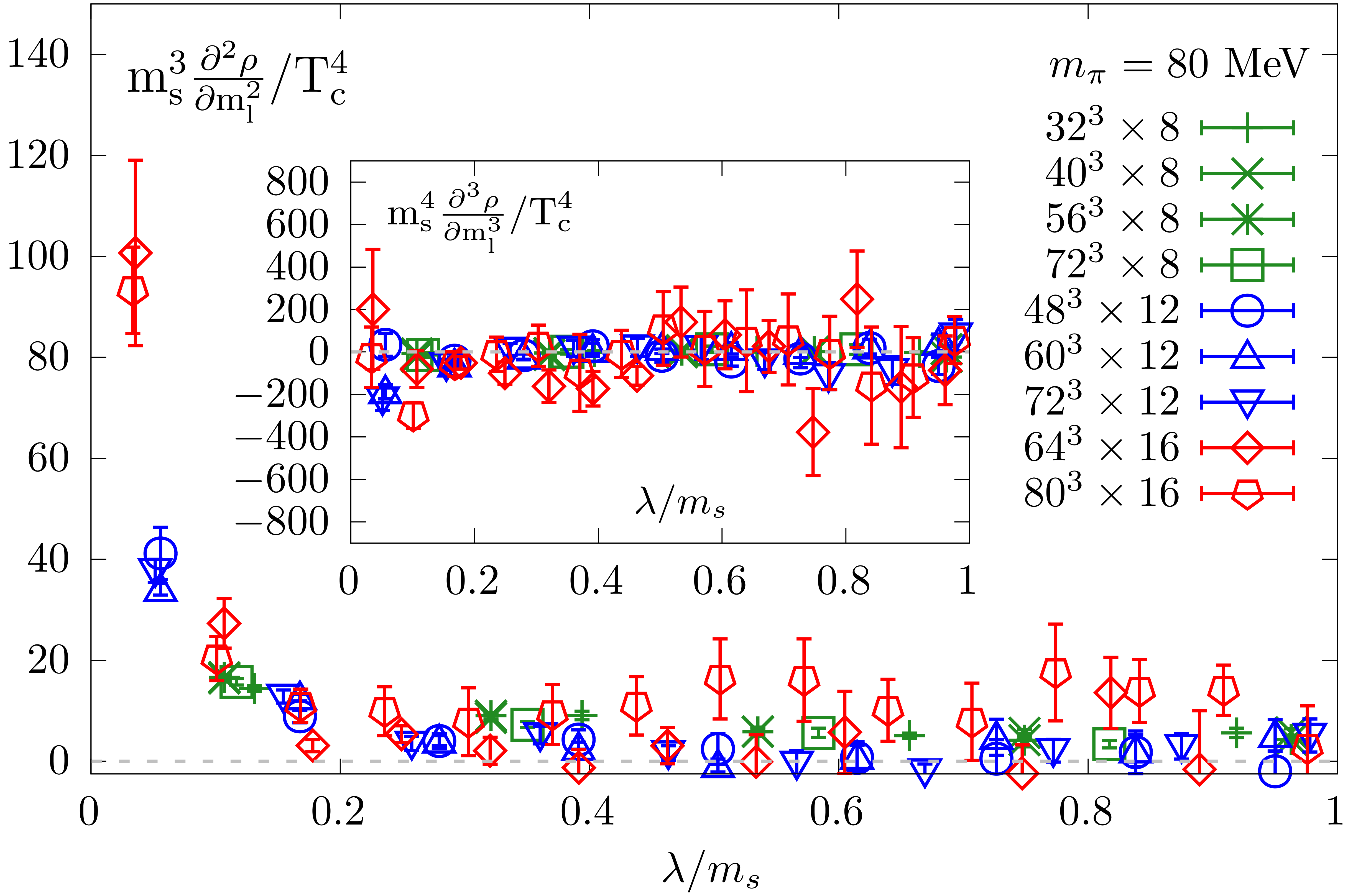
$$m_l^{-1}\partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2$$

Quark mass independent

Peaked structure developed in the small λ region

Drops rapidly towards zero for λ/T >1

2nd & 3rd mass derivative of ρ: volume and a dependences



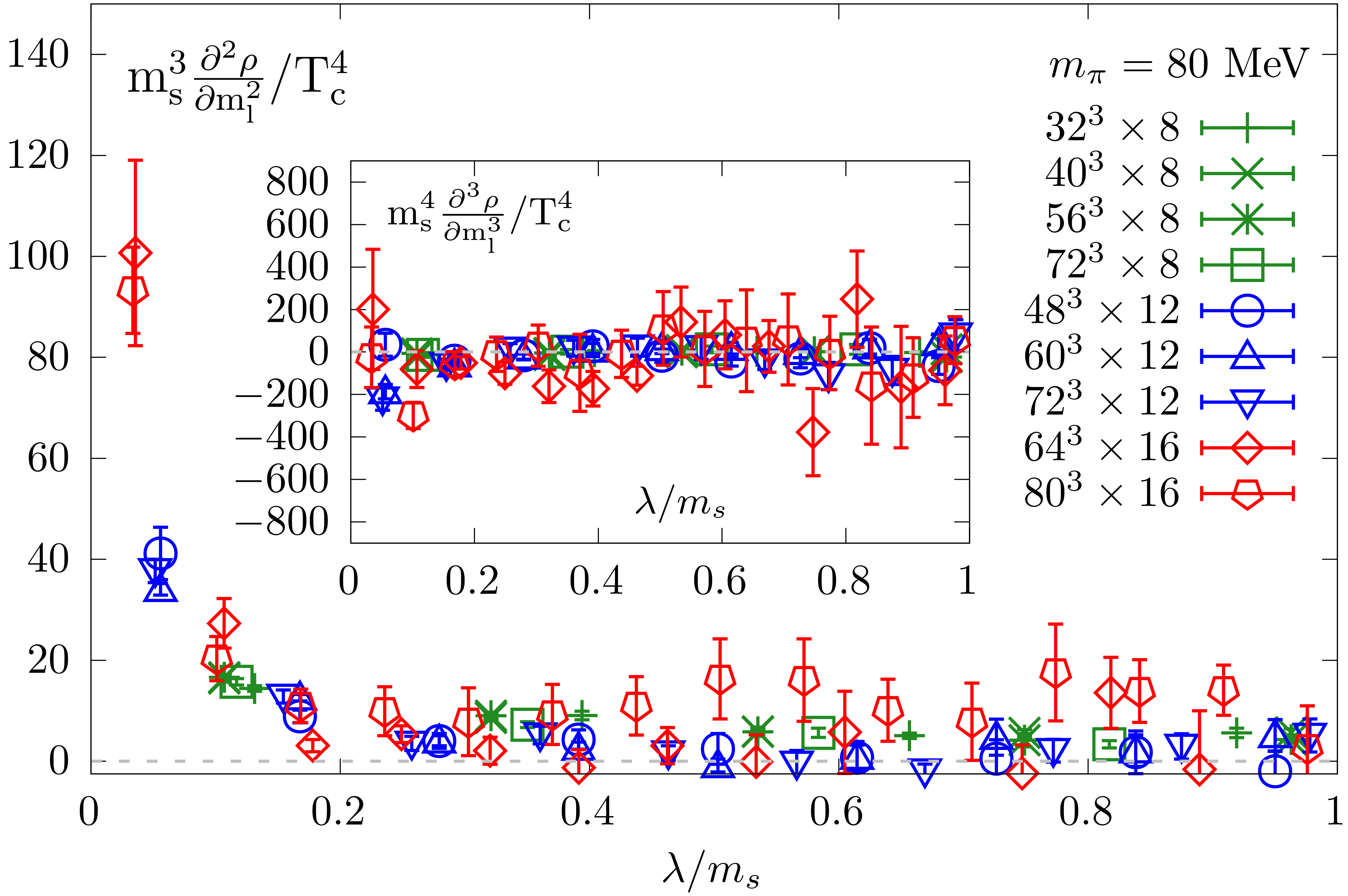
T_c=132 MeV is used from
HTD et al, [HotQCD] PRL 19'

Peaked structure becomes sharper towards continuum limit

Mild volume dependence

$$\partial^3\rho/\partial m_l^3 \approx 0$$

2nd & 3rd mass derivative of ρ: volume and a dependences



T$_c$=132 MeV is used from
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Peaked structure becomes sharper towards continuum limit

Mild volume dependence

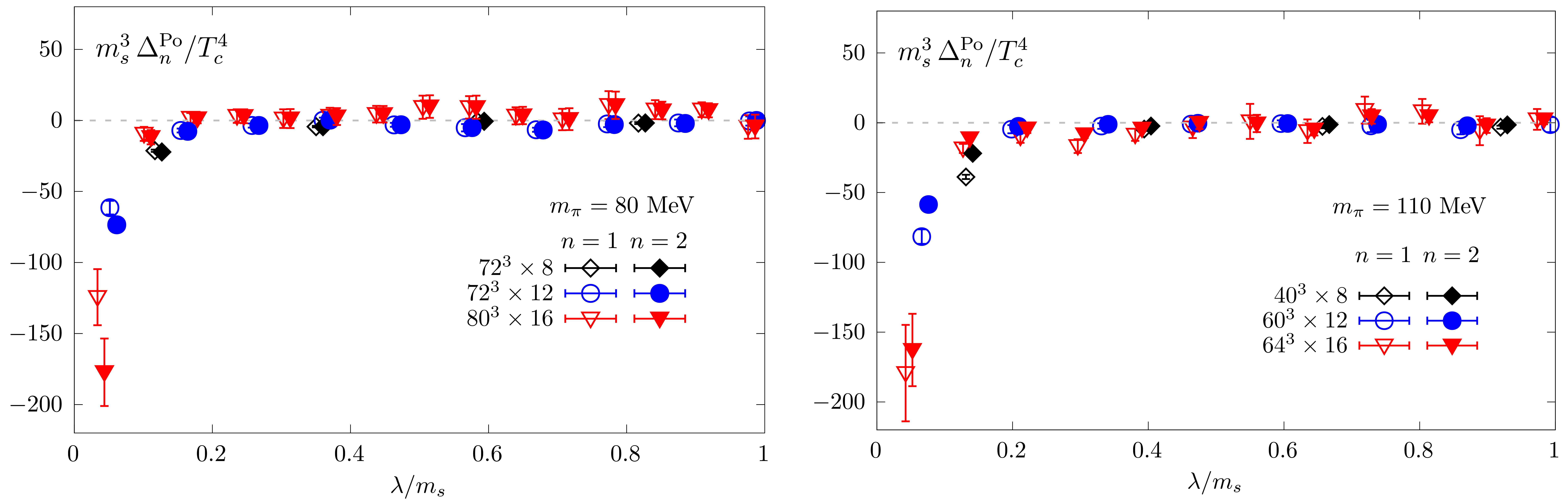
$$\partial^3\rho/\partial m_l^3 \approx 0$$

$$m_l^{-1}\partial\rho/\partial m_l \approx \partial^2\rho/\partial m_l^2$$

$$\rho(\lambda \to 0, m_l \to 0) \propto m_l^2$$

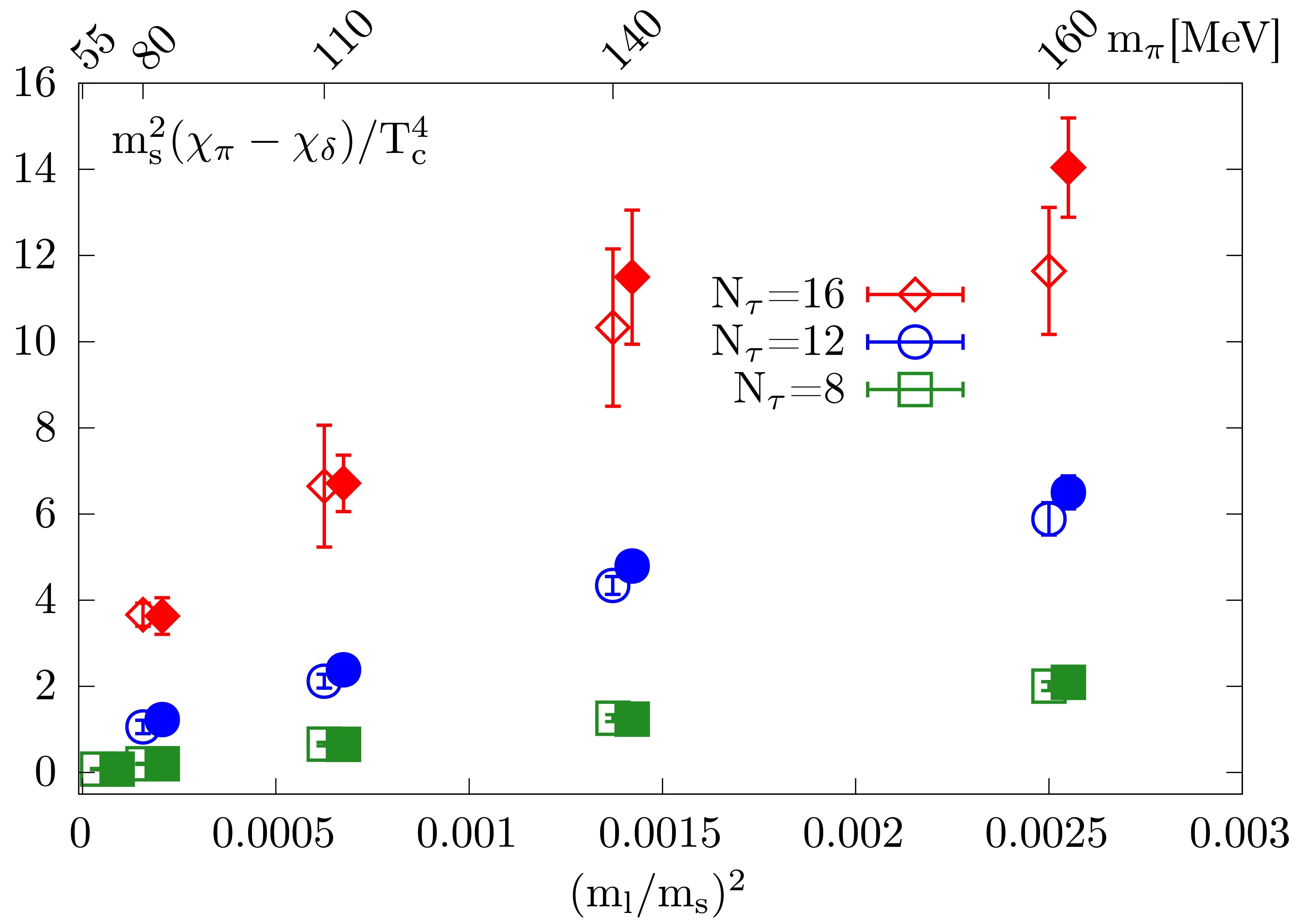
Non-Poisson correlations

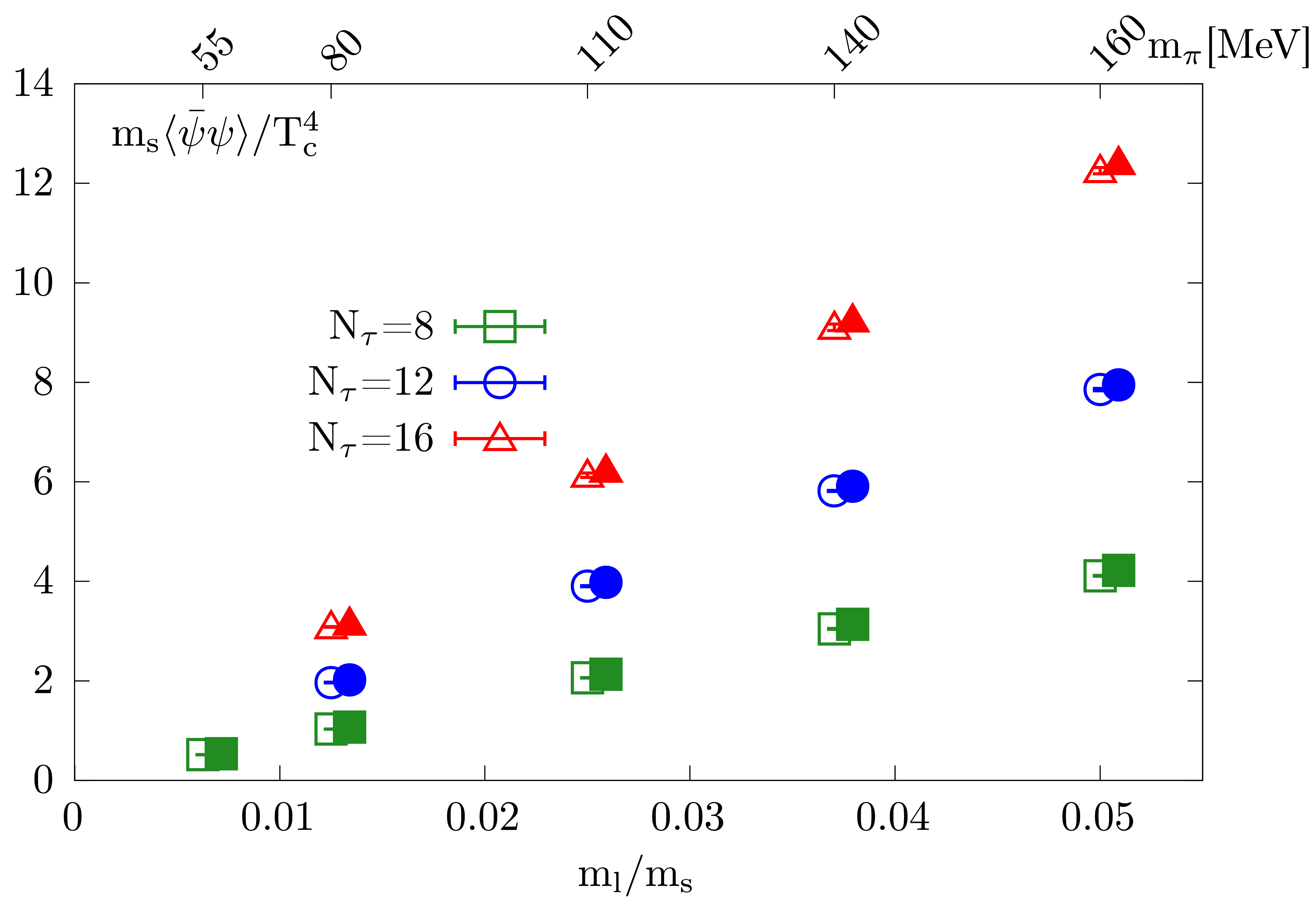
$$\Delta_n^{\mathrm{Po}} = m_l^{n-2}\left[\partial^n \rho/\partial m_l^n - (\partial^n \rho/\partial m_l^n)^{\mathrm{Po}}\right]$$



Repulsive non-Poisson correlation gives rise to the $\rho(\lambda \to 0)$ peak

Quantities related to ρ

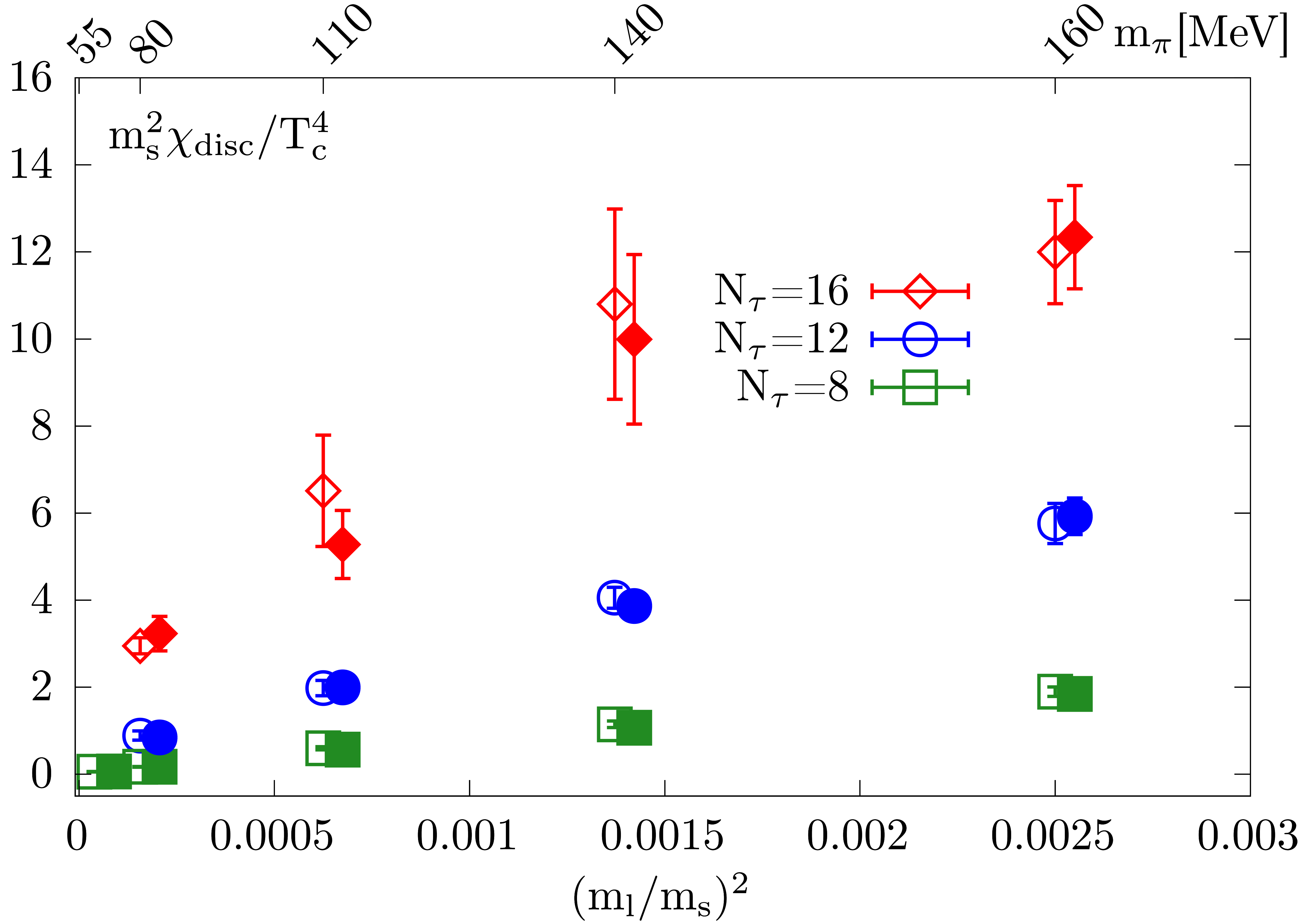


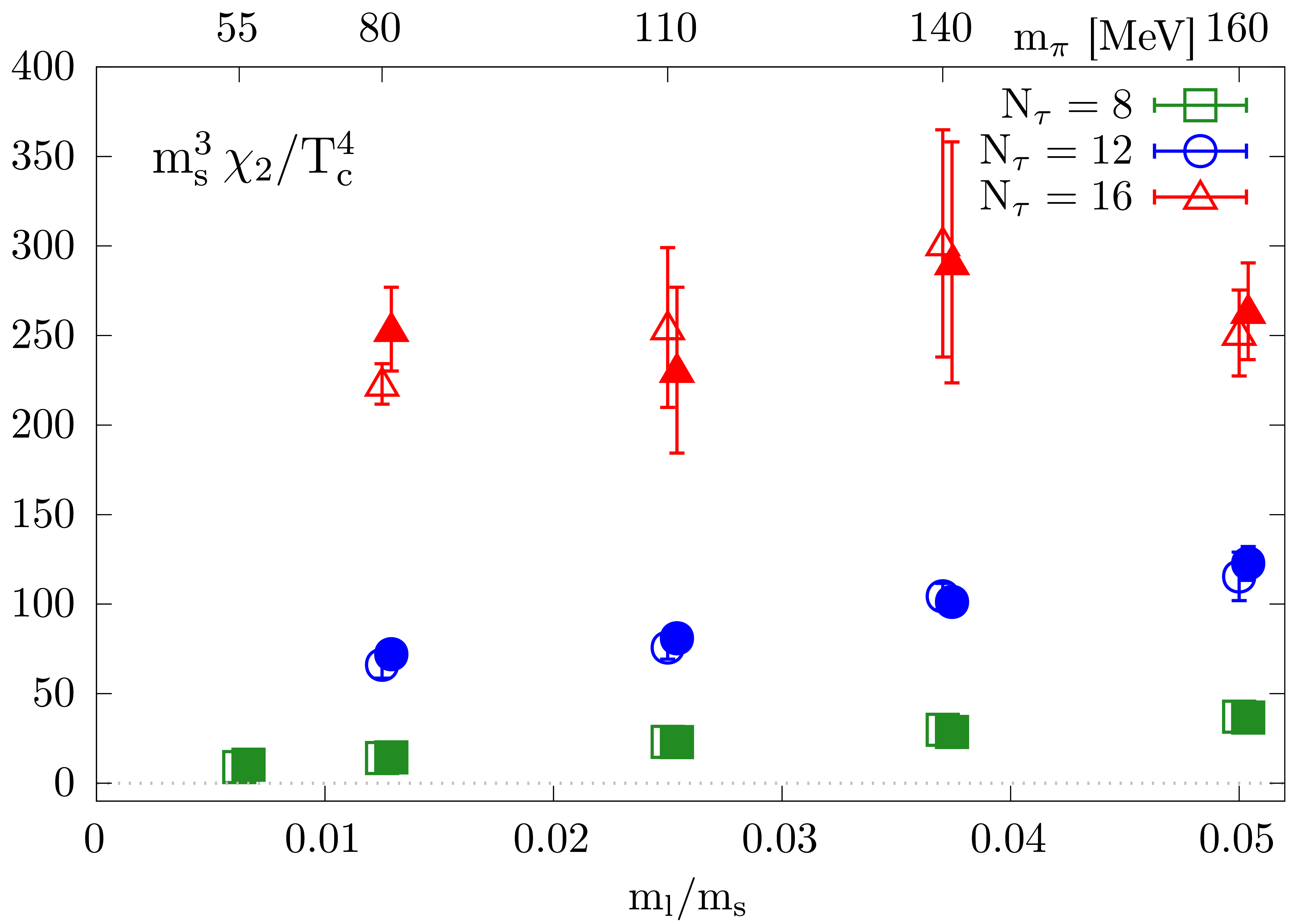


$$\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \, \frac{8m_l^2 \rho}{(\lambda^2 + m_l^2)^2}$$

$$\langle\bar\psi\psi\rangle = \int_0^\infty \frac{4m_l \rho}{\lambda^2 + m_l^2} d\lambda$$

Quantities related to 1st & 2nd derivatives of ρ

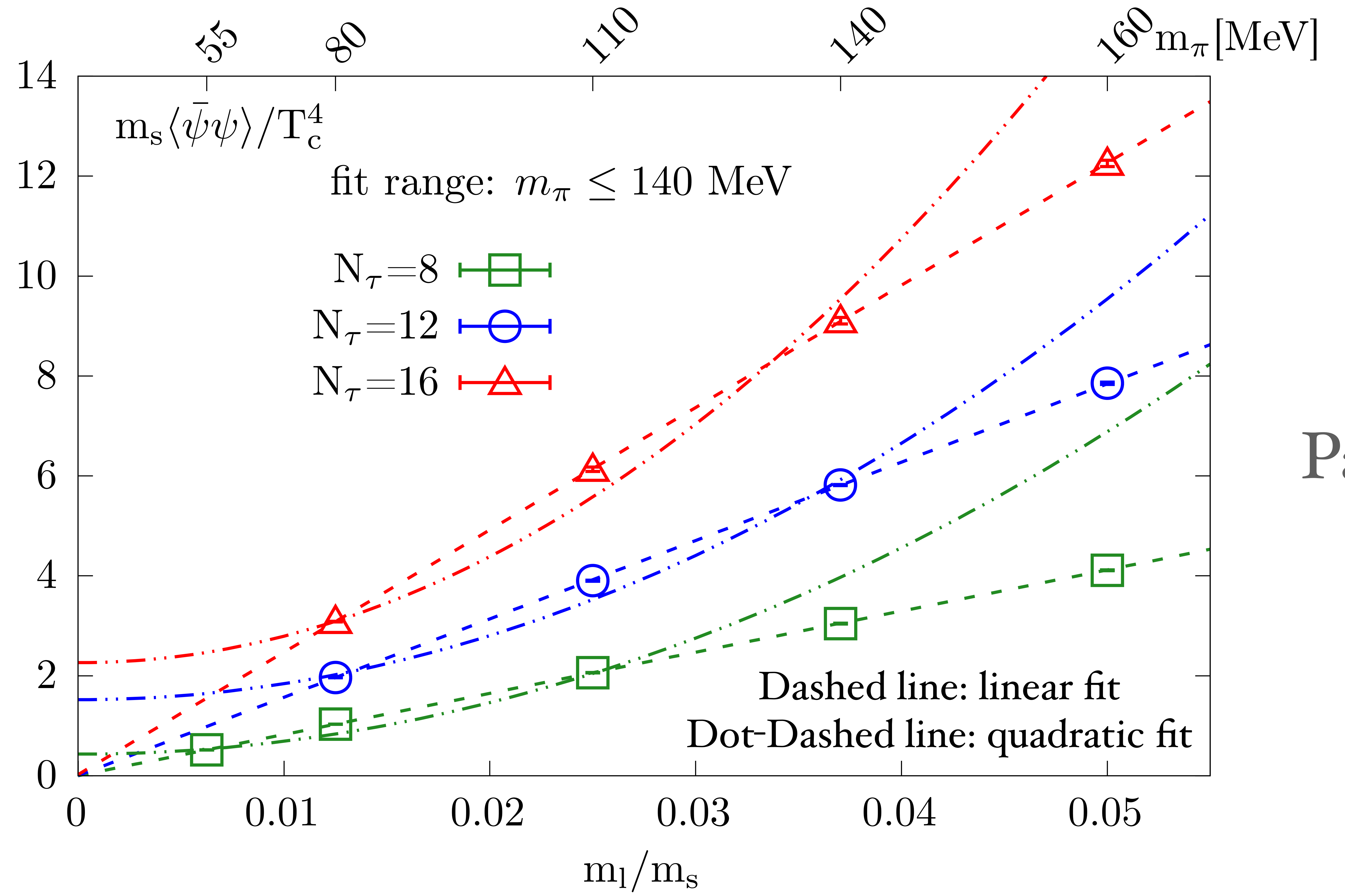




$$\chi_{\rm disc} = \int_0^\infty d\lambda \, \frac{4m_l \, \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

$$\chi_2 = \int_0^\infty d\lambda \, \frac{4m_l \, \partial^2\rho/\partial m_l^2}{\lambda^2 + m_l^2}$$

SU(2)xSU(2) symmetry restoration at T=205 MeV



χ^2/dof	Linear fits	Quadratic fits
N_τ=8	0.43	13972.7
N_τ=12	4.4	1504.0
N_τ=16	0.1	198.5

In the chiral symmetric phase

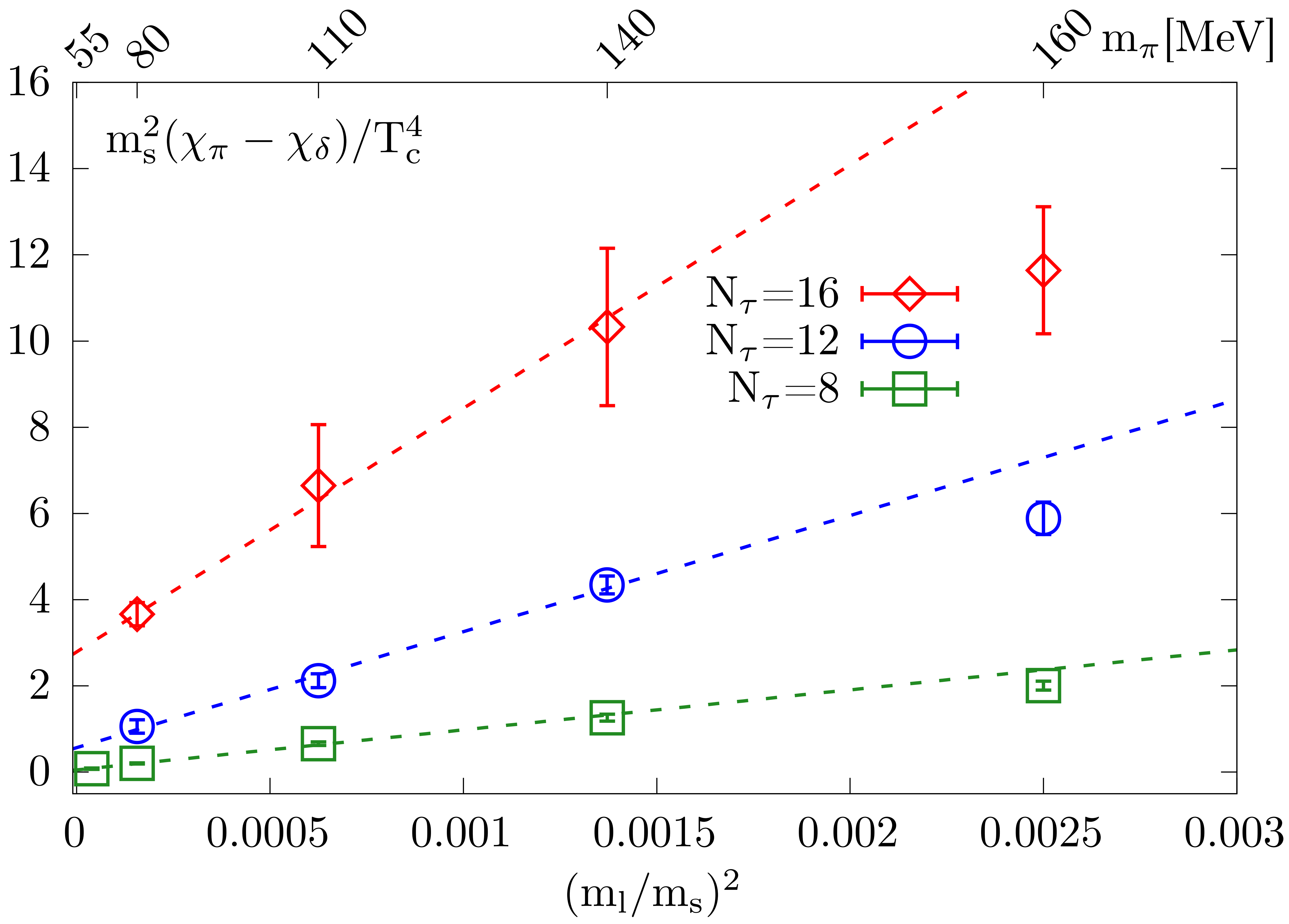
Z(2) subgroup of SU(2)xSU(2) sym.

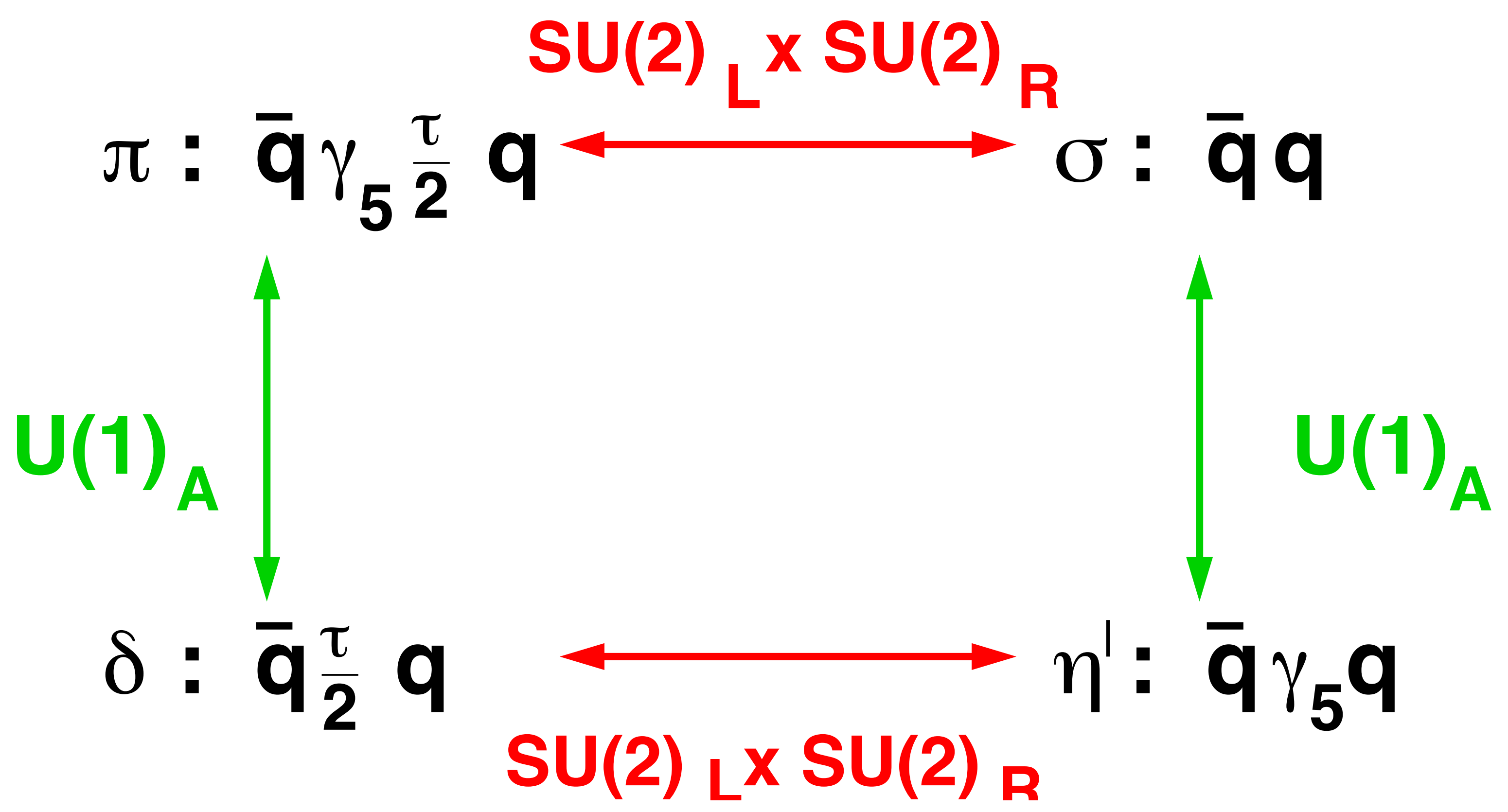
Partition function: even function of m

$$\langle\bar{\psi}\psi\rangle \propto m \text{ as } m \to 0$$

$$\chi_{disc} \propto m^2 \text{ as } m \to 0$$

Difference between π and δ susceptibilities





- Linear behavior in quark mass squared at $m_\pi \leq 140$ MeV
- Linear fits wo m_π =160 MeV data at each Nt yield values at m=0:

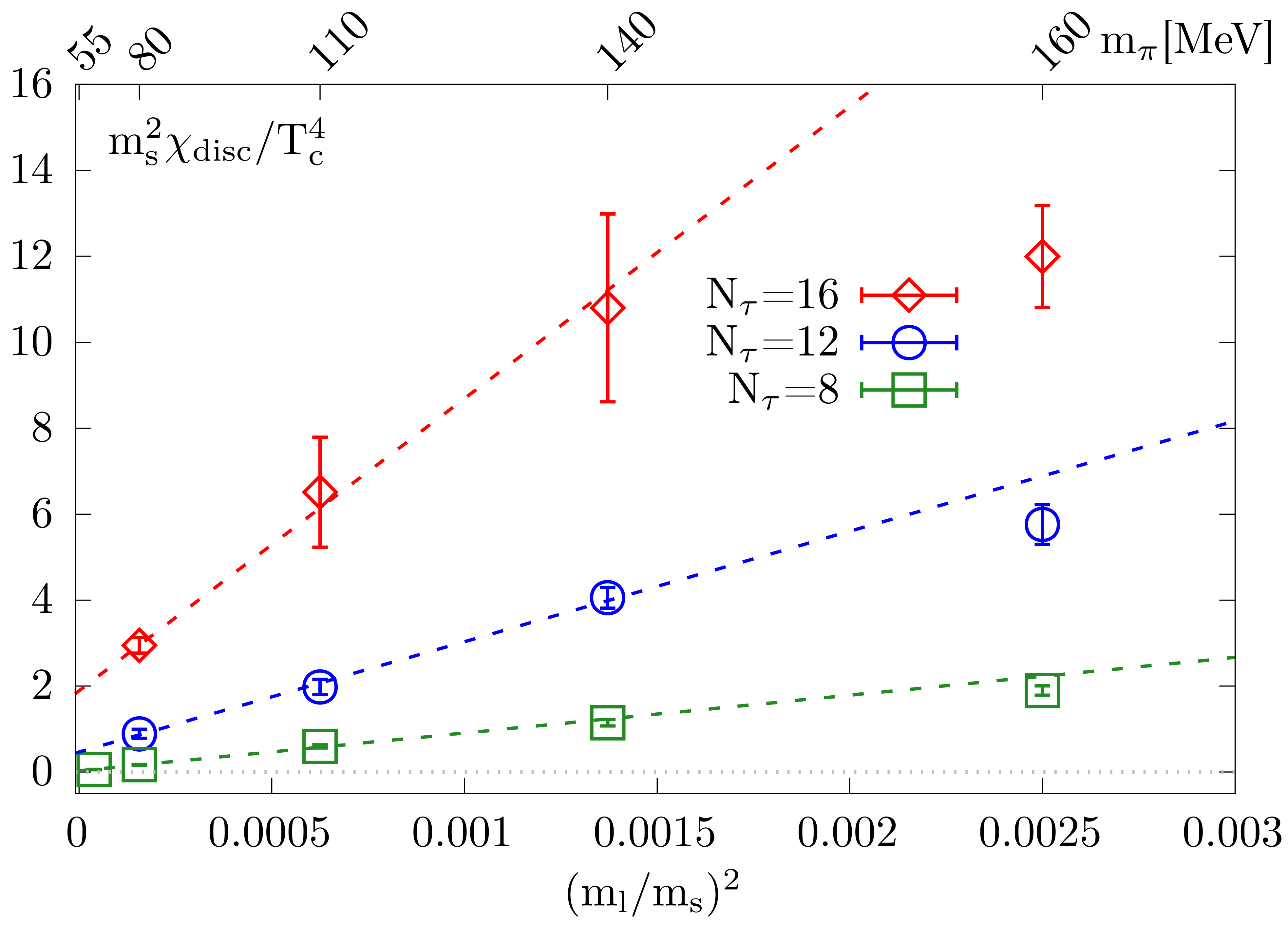
N_τ=8: 0.05(1)

N_τ=12: 0.6(2)

N_τ=16: 2.8(1)

Two $U(1)_A$ measures

$\chi_\pi - \chi_\delta$ should equal to χ_{disc} in chiral symmetric QCD



N_τ=8: 0.0030(7)

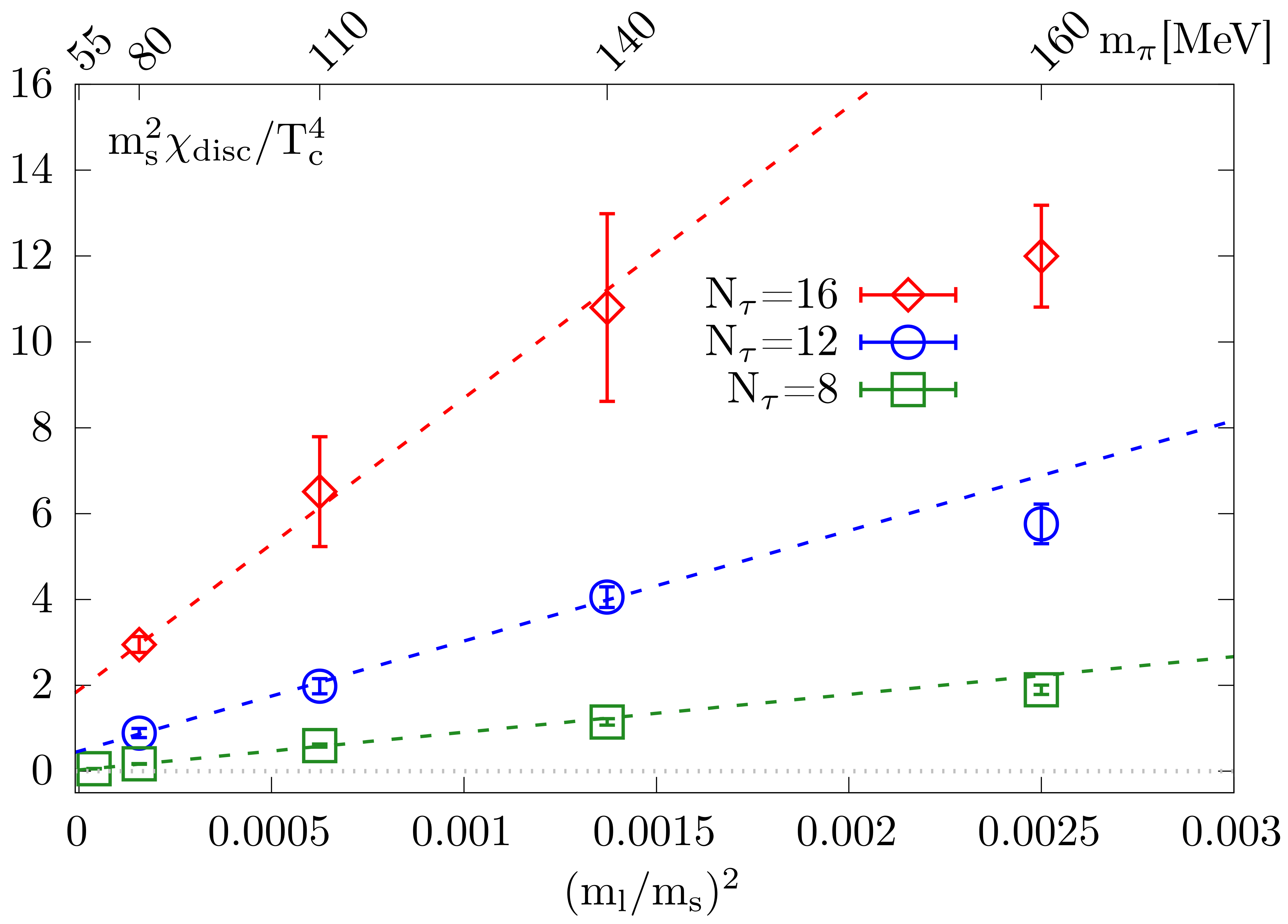
N_τ=12: 0.47(8)

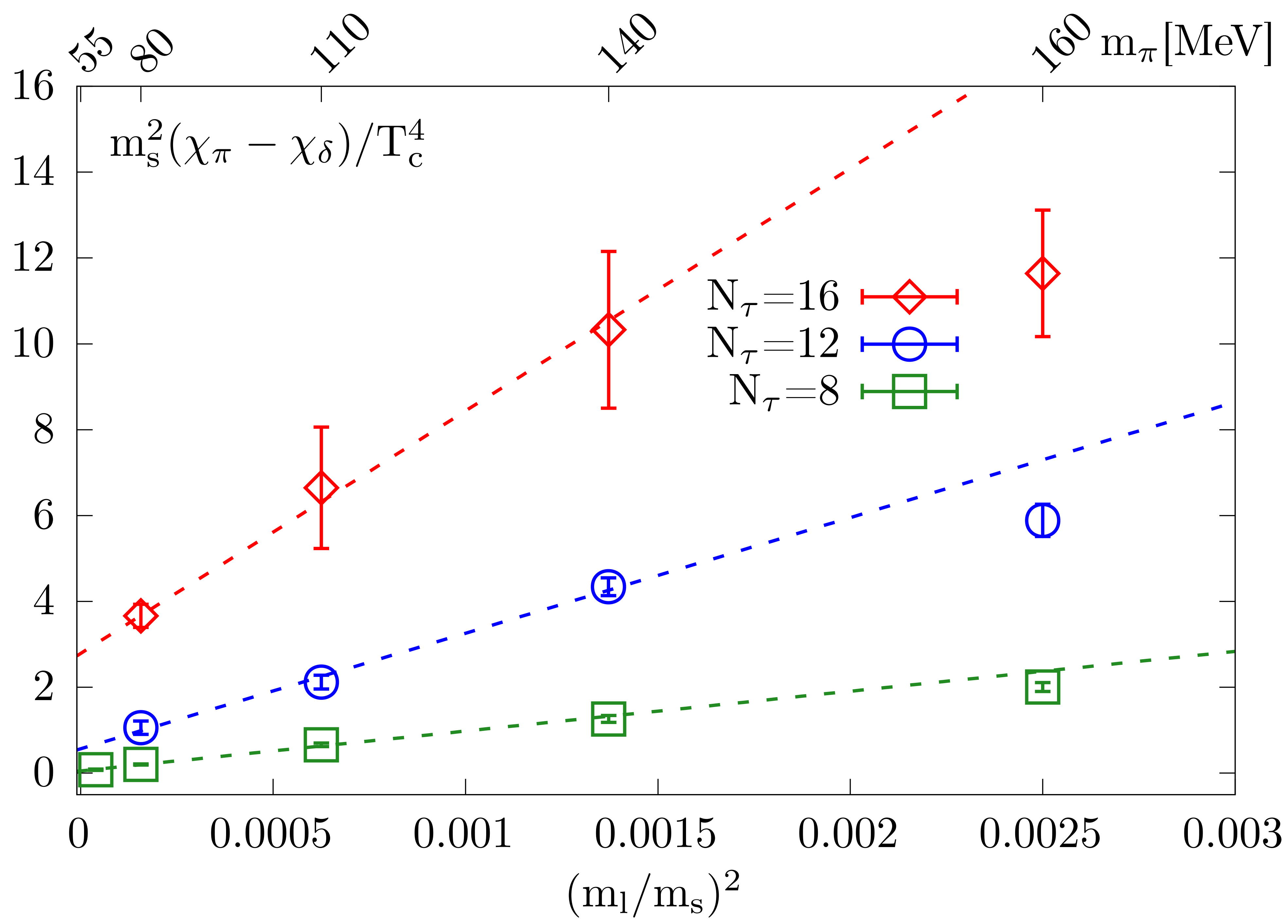
N_τ=16: 1.9(1)

Values in the chiral limit at each N_τ

Two U(1)A measures

$\chi_\pi - \chi_\delta$ should equal to χ_{disc} in chiral symmetric QCD





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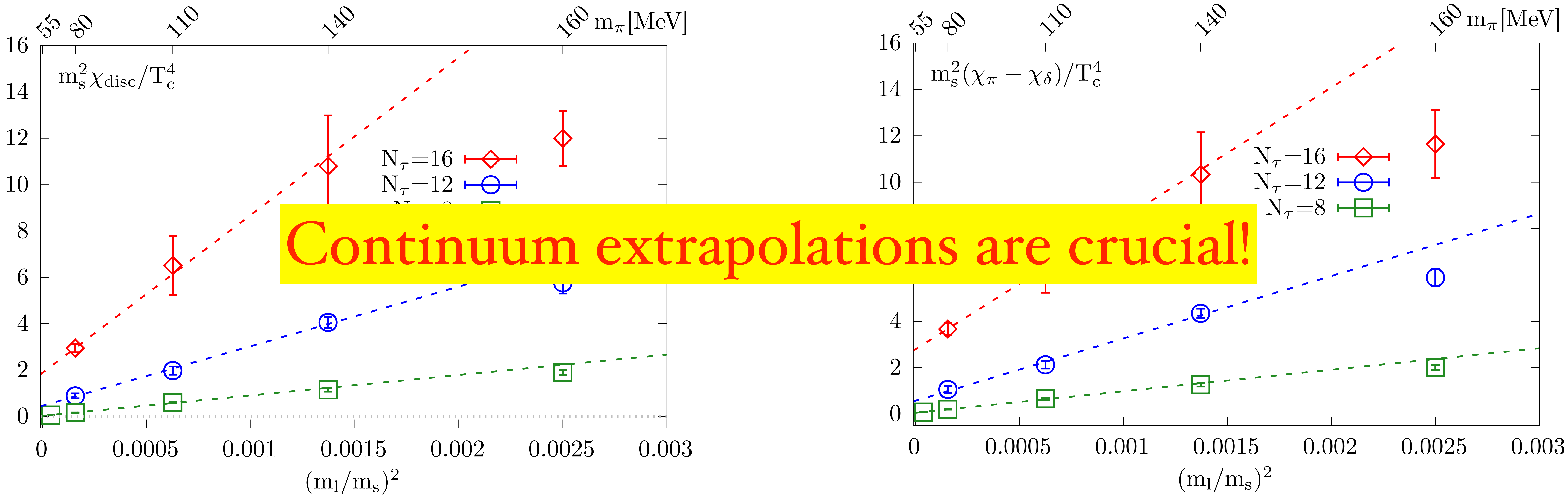

N_τ=8: 0.0030(7)
N_τ=12: 0.47(8)
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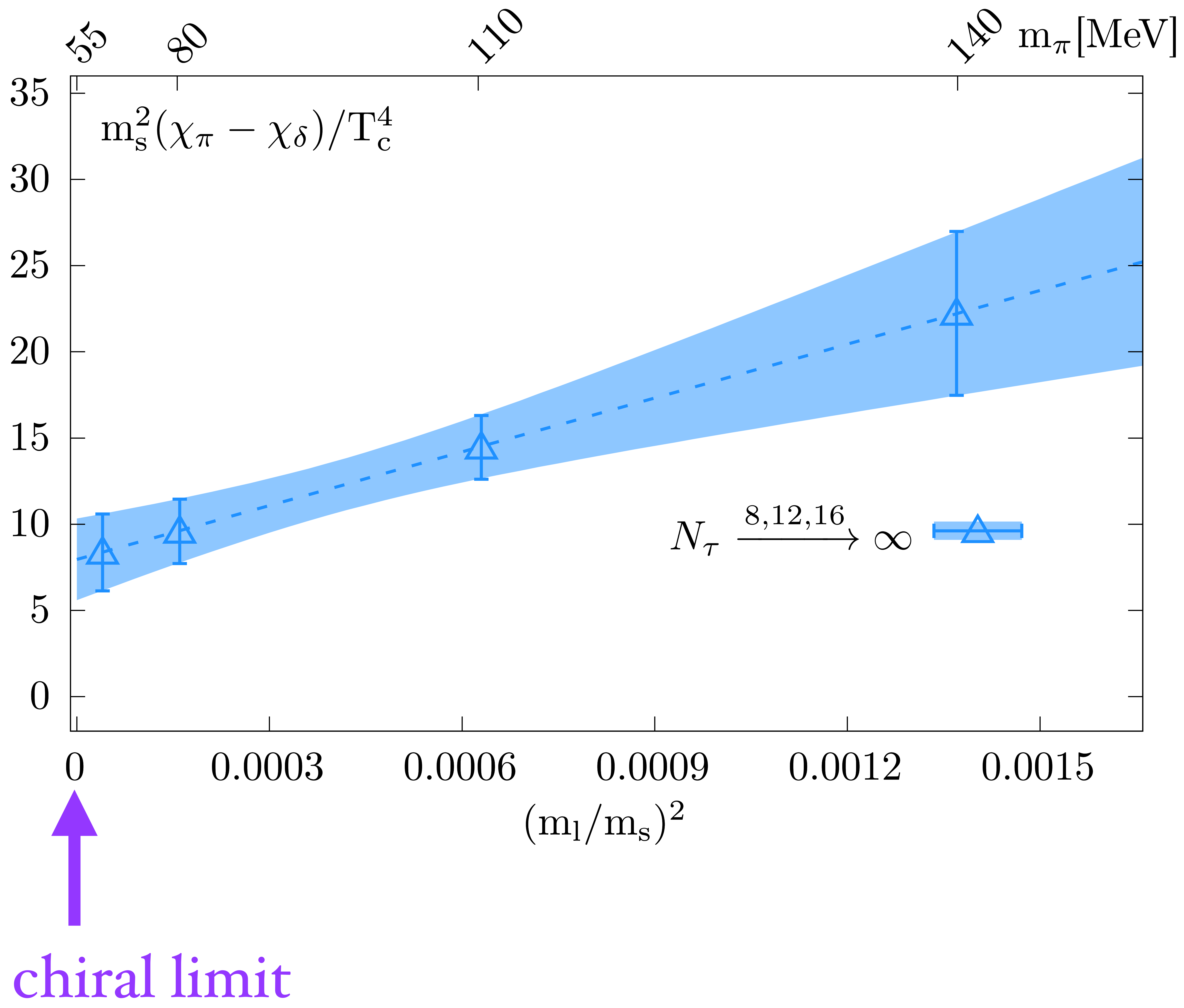
Values in the chiral limit at each N_τ

N_τ=8: 0.05(1)
N_τ=12: 0.6(2)
N_τ=16: 2.8(1)

Continuum and chiral extrapolations with $m_\pi \leq 140$ MeV data



Joint fit: simultaneous fits

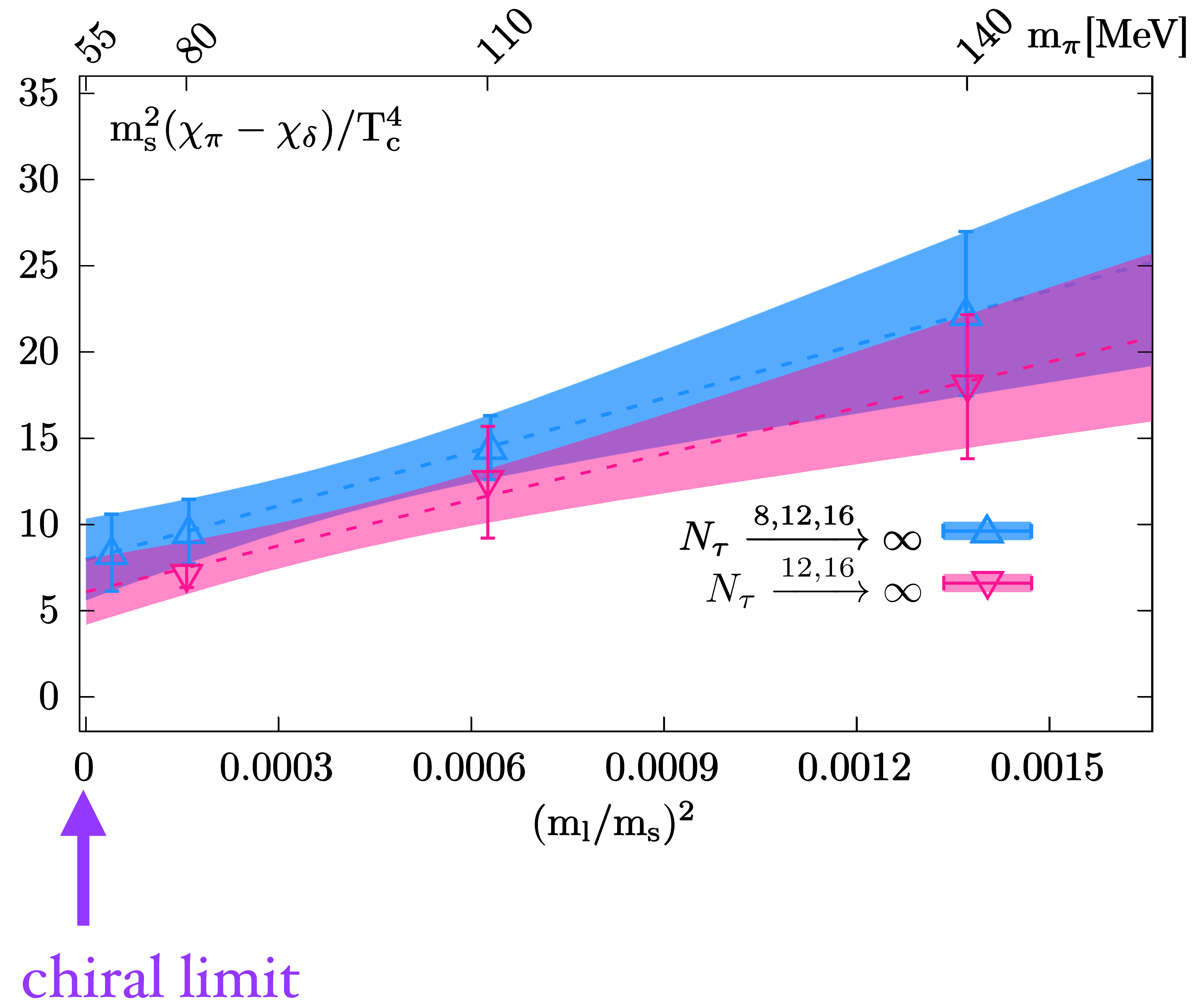
Continuum: quadratic in $1/N_\tau$

Chiral: quadratic in quark mass

Value at $N_\tau \to \infty$ and $m \to 0$:

8.0 ± 2.4

Continuum and chiral extrapolations with $m_\pi \leq 140$ MeV data



- **Joint fit**: simultaneous fits

  Continuum: quadratic in $1/N_\tau$

  Chiral: quadratic in quark mass

  Value at $N_\tau \to \infty$ and $m \to 0$:

  8.0 ± 2.4

- **Sequential fit:** first continuum and then chiral extrapo.

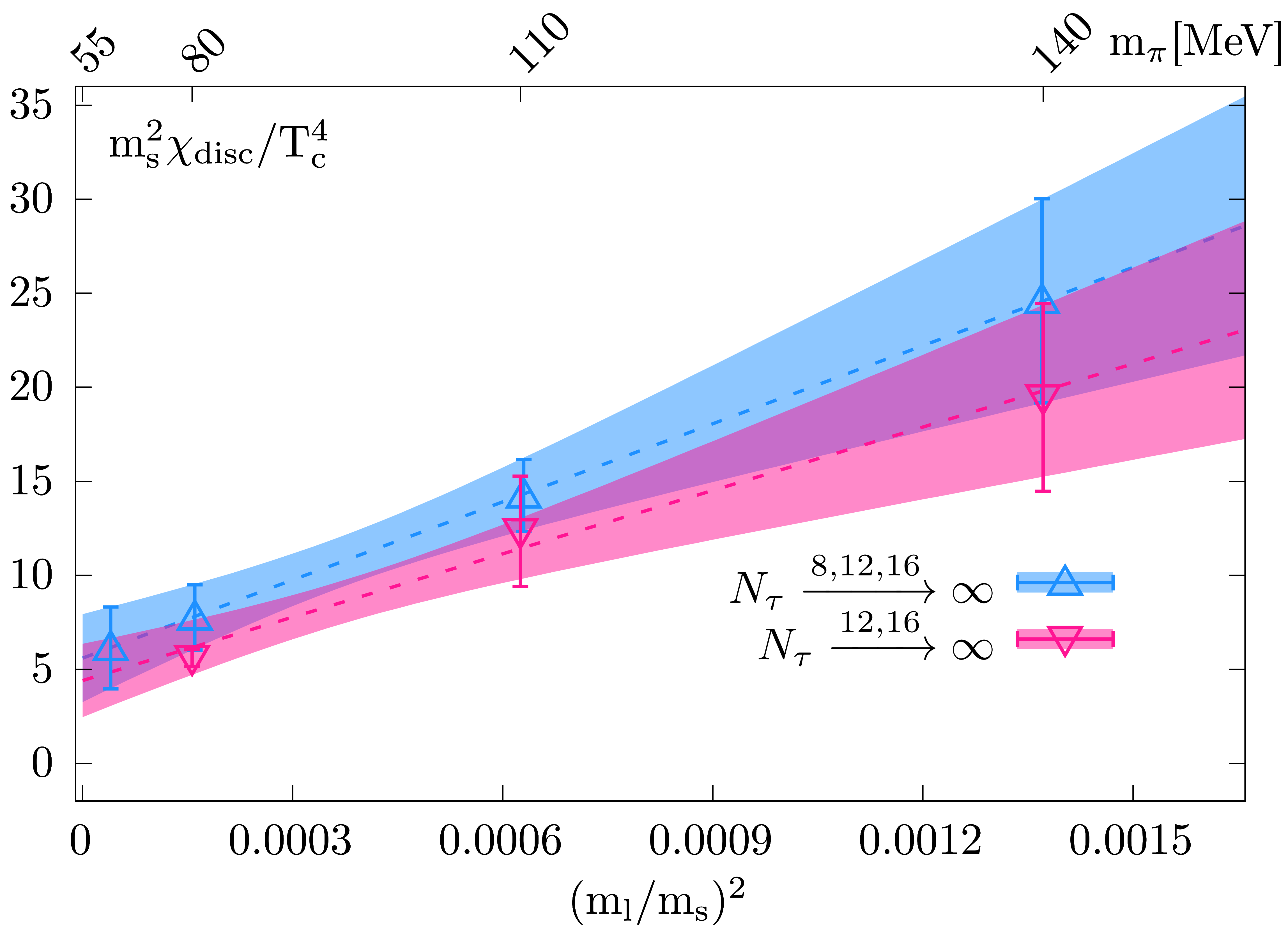
  Continuum: quadratic in $1/N_\tau$ with N_τ=12 & 16 data

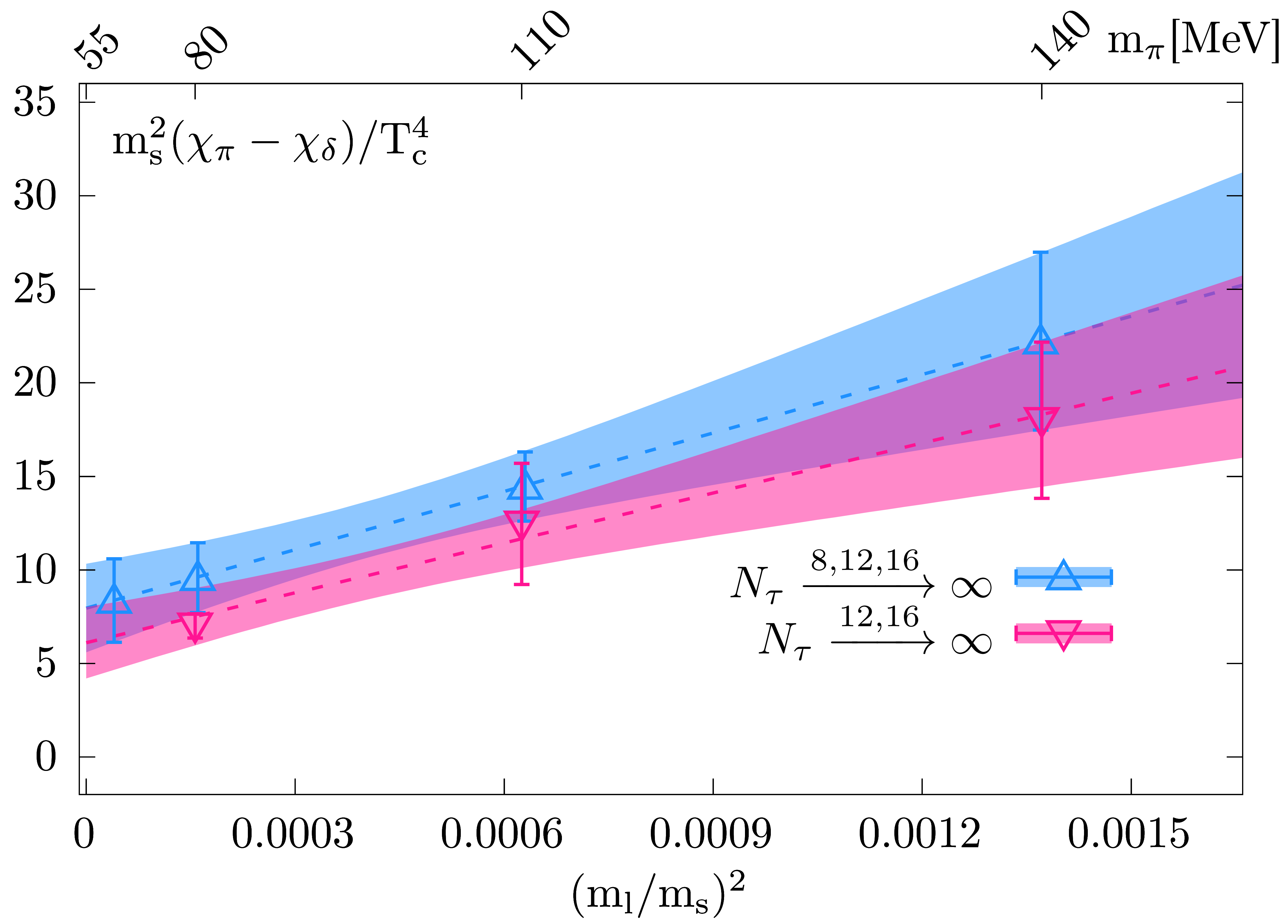
  Chiral: quadratic in quark mass

  Value at $N_\tau \to \infty$ and $m \to 0$:

  6.1 ± 1.9

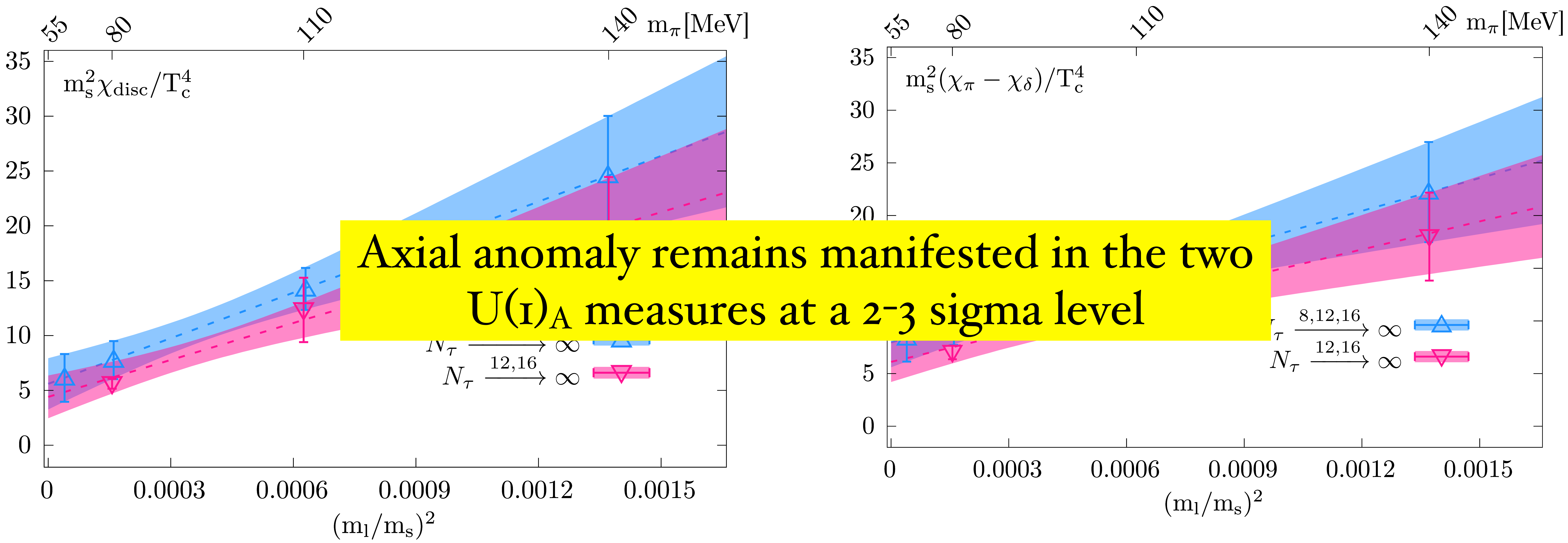
Continuum and chiral extrapolations with $m_\pi \leq 140$ MeV data





$N_\tau \to \infty$ and $m \to 0$	$m_s^2 \chi_{disc}/T_c^4$	$m_s^2(\chi_\pi - \chi_\delta)/T_c^4$
Joint fit	**5.6 ± 2.3**	**8.0 ± 2.4**
Sequential fit	**4.4 ± 1.9**	**6.1 ± 1.9**

Continuum and chiral extrapolations with $m_\pi \leq 140$ MeV data



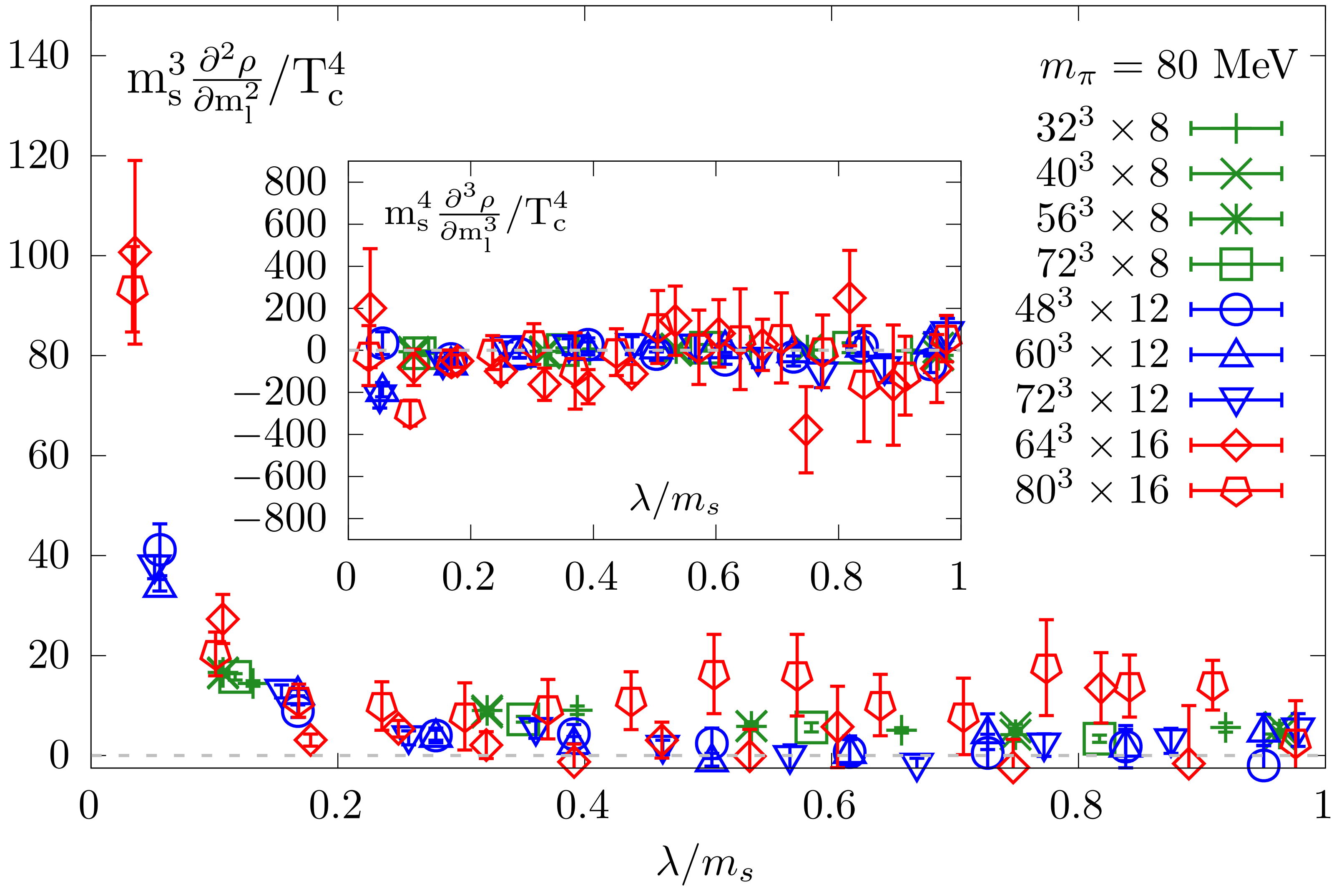
Axial anomaly remains manifested in the two $U(1)_A$ measures at a 2-3 sigma level

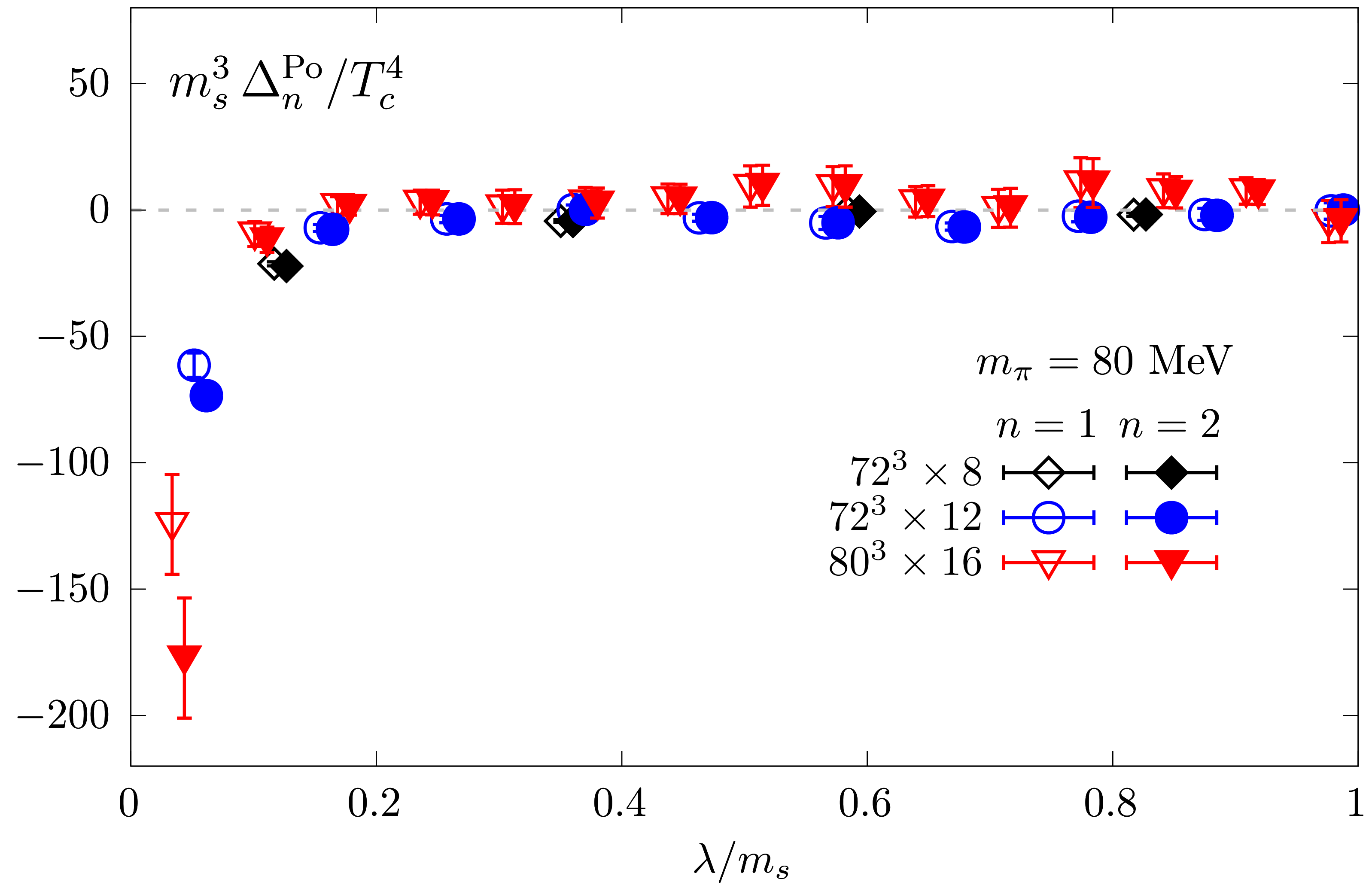
$N_\tau \to \infty$ and $m \to 0$	$m_s^2\chi_{disc}/T_c^4$	$m_s^2(\chi_\pi - \chi_\delta)/T_c^4$
Joint fit	**5.6 ± 2.3**	**8.0 ± 2.4**
Sequential fit	**4.4 ± 1.9**	**6.1 ± 1.9**

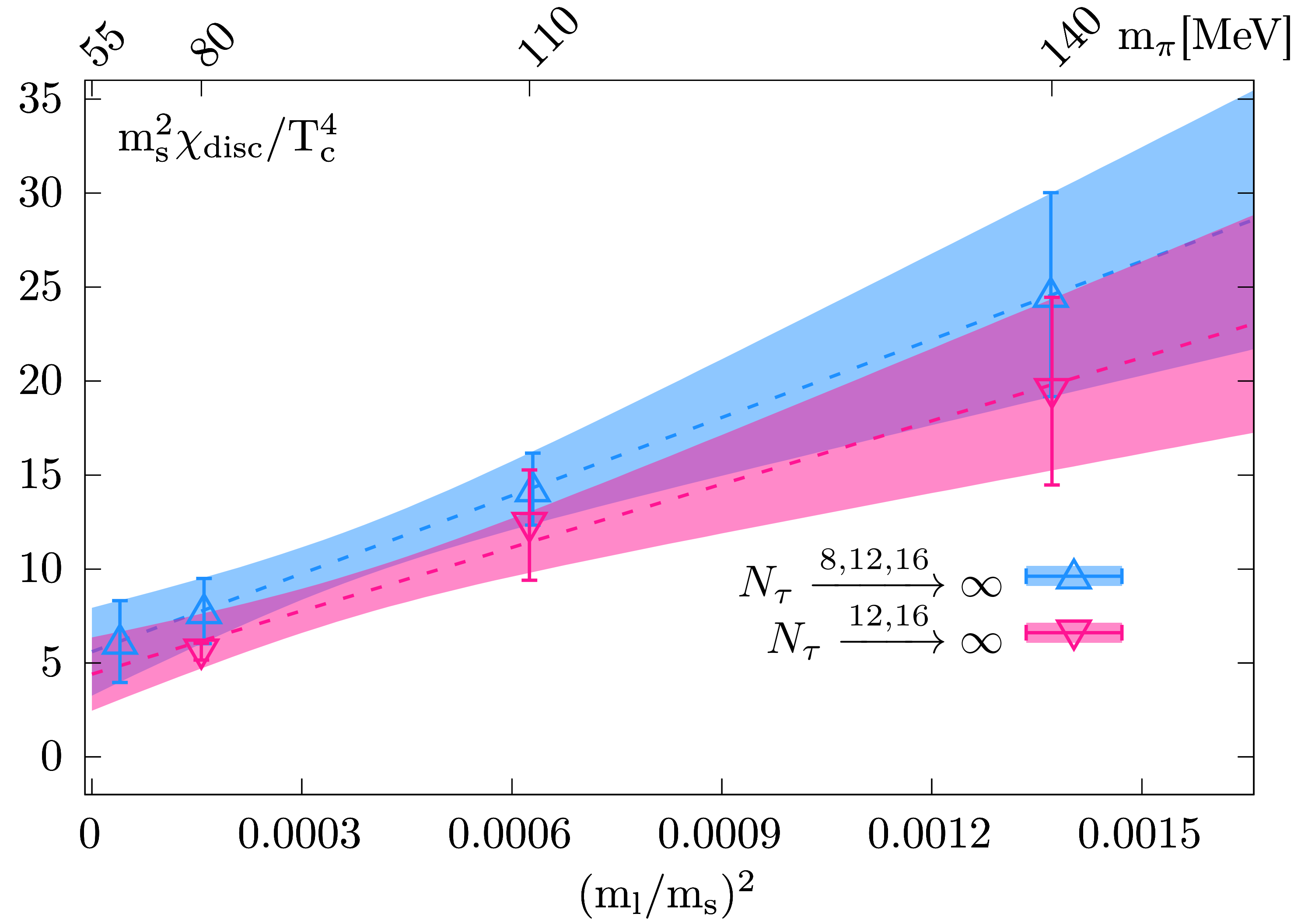
Summary & Conclusion

- We established novel relations between $\partial^n \rho/\partial m^n$ & C_{n+1}

In (2+1)-flavor QCD at $T \approx 1.6\, T_c$







Summary & Conclusion

Our study suggests:

- At $T \gtrsim 1.6\,T_c$ the microscopic origin of axial anomaly is driven by the weakly interacting (quasi-) instanton gas motivated $\rho(\lambda \to 0, m \to 0) \propto m^2\, \delta(\lambda)$

- N_f=2+1 QCD: 2nd order chiral phase transition belonging to 3-d O(4)

Outlook:

- [ ] the methodology would be useful for other discretization schemes