Energy, momentum and stresses in hadrons

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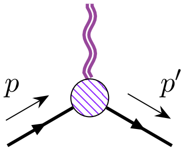
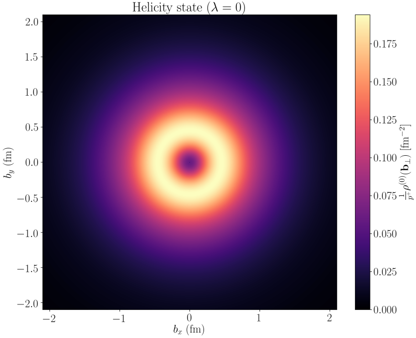
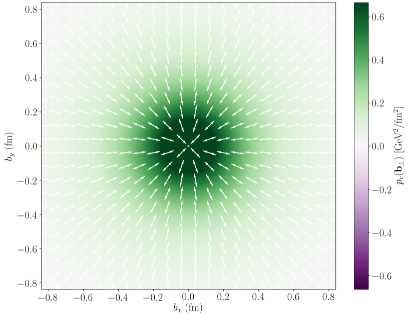
Introduction

- The **energy-momentum tensor** (EMT) is an operator characterizing the distribution and flow of energy and momentum.
- Matrix elements between hadronic states characterize coveted properties of hadrons:
  - The distribution & decomposition of energy.
  - The distribution & decomposition of momentum.
  - The distribution & decomposition of internal stresses







The canonical EMT

- **Canonical** EMT: via Noether's first theorem & global spacetime translations.

$$\hat{T}^{\mu\nu}_{\text{can.}}(x) = \sum_q \frac{\delta \mathcal{L}}{\delta \partial_\mu q(x)} \partial^\mu q(x) + \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\lambda(x)} \partial^\nu A_\lambda(x) - g^{\mu\nu} \mathcal{L}$$

$$= \sum_q \left\{ \bar{q}(x) i \gamma^\mu \overleftrightarrow{D}^\nu q(x) \right\} - 2\mathrm{Tr}\left[G^{\mu\lambda} \partial^\nu A_\lambda \right] + \frac{1}{2} g^{\mu\nu} \mathrm{Tr}\left[G^{\lambda\sigma} G_{\lambda\sigma} \right]$$

- $x^\mu \mapsto x^\mu + \xi^\mu$
- $S_{\text{QCD}} \mapsto S_{\text{QCD}}$
- $\partial_\mu T^{\mu\nu}_{\text{can}} = 0.$
- Mot symmetric: $T^{\mu\nu}_{\text{can.}} \neq T^{\nu\mu}_{\text{can.}}$
- **Not gauge-invariant.**

The Belinfante-Rosenfeld EMT

- **Belinfante** EMT: add trivially conserved "superpotential" to canonical EMT.

$$\hat{T}^{\mu\nu}_{\mathrm{Bel}}(x) = \sum_q \left\{ \frac{1}{2}\bar{q}(x) i\gamma^{\{\mu} \overleftrightarrow{D}^{\nu\}} q(x) \right\} - 2\mathrm{Tr}\left[G^{\mu\lambda} G^{\nu}{}_{\lambda} \right] + \frac{1}{2} g^{\mu\nu} \mathrm{Tr}\left[G^{\lambda\sigma} G_{\lambda\sigma} \right]$$

  - $T^{\mu\nu}_{\mathrm{Bel}} = T^{\mu\nu}_{\mathrm{can}} + \partial_\sigma \Lambda^{\sigma\mu\nu}$.
  - $\Lambda^{\sigma\mu\nu} = -\Lambda^{\mu\sigma\nu}$
  - $\partial_\mu T^{\mu\nu}_{\mathrm{Bel}} = 0$.
  - Symmetric: $T^{\mu\nu}_{\mathrm{Bel}} = T^{\nu\mu}_{\mathrm{Bel}}$
  - **Gauge-invariant.**
- Any $\Lambda^{\sigma\mu\nu} = -\Lambda^{\mu\sigma\nu}$ produces a "valid" EMT—why this one?

The Hilbert EMT

- **Hilbert EMT**: functional derivative of action with respect to metric:

$$\hat{T}^{\mu\nu}_{\text{Hil}}(x) = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{QCD}}}{\delta g_{\mu\nu}(x)}$$

$$= \sum_q \left\{ \frac{1}{2}\bar{q}(x) i\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}} q(x) \right\} - 2\text{Tr}\left[G^{\mu\lambda}G^{\nu}{}_{\lambda}\right] + \frac{1}{2}g^{\mu\nu}\text{Tr}\left[G^{\lambda\sigma}G_{\lambda\sigma}\right]$$

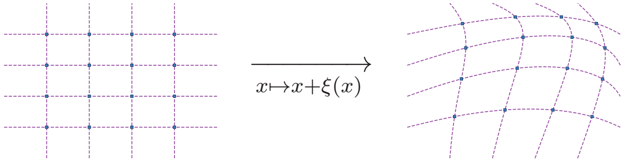
  - $\partial_\mu T^{\mu\nu}_{\text{Hil}} = 0$.
  - Symmetric: $T^{\mu\nu}_{\text{Hil}} = T^{\nu\mu}_{\text{Hil}}$
  - **Gauge-invariant.**
- In Belinfante procedure, can choose $\Lambda^{\sigma\mu\nu}$ to give Hilbert EMT.
- Why use Noether's (first) theorem if Hilbert procedure gives the right EMT?
  - Transformation properties of separate fields allow **energy & spin decomposition**.
  - But what's the decomposition of the added terms?

Noether's theorems revisited

- Conserved current from *local* spacetime translations (**Noether's second theorem**):

$$\hat{T}^{\mu\nu}_{\mathrm{QCD}}(x) = \sum_q \left\{ \frac{1}{2}\bar{q}(x) i\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}} q(x) \right\} - 2\mathrm{Tr}\left[G^{\mu\lambda}G^{\nu}{}_{\lambda}\right] + \frac{1}{2}g^{\mu\nu}\mathrm{Tr}\left[G^{\lambda\sigma}G_{\lambda\sigma}\right]$$

  - $x^\mu \mapsto x^\mu + \xi^\mu(x)$
  - $S_{\mathrm{QCD}} \mapsto S_{\mathrm{QCD}}$
  - $\partial_\mu T^{\mu\nu}_{\mathrm{QCD}} = 0.$
  - $T^{\mu\nu}_{\mathrm{QCD}} = T^{\nu\mu}_{\mathrm{QCD}}$
  - **Gauge-invariant**



AF, Phys. Rev. D 106 (2022) 125012

- Guaranteed to coincide with Hilbert EMT—equivalent to transforming metric.
  - Transformation properties of separate fields allow **energy & spin decomposition**.

    (future work!)

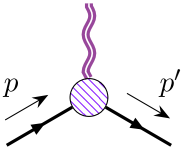
Form factors of the EMT

- EMT matrix elements give **gravitational form factors** (GFFs).
  - It's just a name.
  - EMT is the source of gravitation: $G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi T^{\mu\nu}$
  - But we don't really use gravitation to measure them.
- Analogy to **electromagnetic form factors**.
- Spin-zero example:

$$\langle p'|\hat{J}^{\mu}(0)|p\rangle = 2P^{\mu}F(t)$$

$$\langle p'|\hat{T}^{\mu\nu}(0)|p\rangle = 2P^{\mu}P^{\nu}A(t) + \frac{1}{2}(\Delta^{\mu}\Delta^{\nu} - \Delta^{2}g^{\mu\nu})D(t)$$

- $A(t)$ encodes momentum density
- $D(t)$ encodes stress distributions (anisotropic pressures)
- Mix of both encodes **energy density**



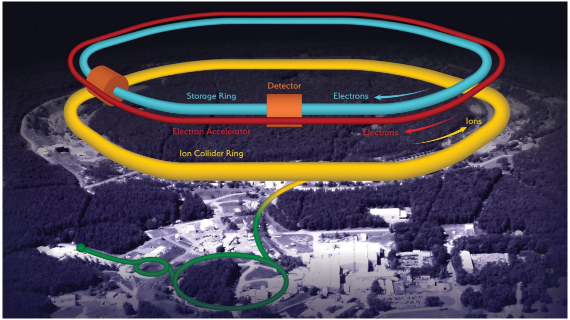
$$P = \frac{1}{2}(p + p')$$

$$\Delta = (p' - p)$$

$$t = \Delta^2$$

How to get the GFFs

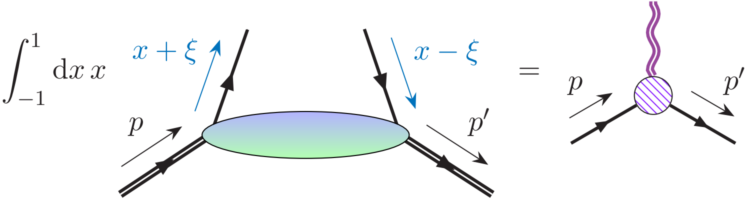
- **Hard exclusive reactions** are used to measure GFFs—not gravity experiments.
  - Deeply virtual Compton scattering (DVCS) to probe quark structure.
  - Deeply virtual meson production (DVMP), *e.g.*, J/ψ or Υ to probe gluon structure.
  - ...and more!
- Measured at **Jefferson Lab** and the upcoming **Electron Ion Collider**.



GFFs and GPDs

- **Hard exclusive reactions** are used to measure GFFs—not gravitational experiments.
  - Deeply virtual Compton scattering (DVCS) to probe quark structure.
  - Deeply virtual meson production (DVMP), *e.g.*, J/ψ or Υ to probe gluon structure.
  - ...and more!
- GFFs are related to **generalized parton distributions** (GPDs) through Mellin moments—spin-zero example:

$$\int_{-1}^{1} \mathrm{d}x\, x H_a(x,\xi,t) = A_a(t) + \xi^2 D_a(t)$$



Components of the EMT

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

- **Momentum densities** (or energy fluxes)
- **Energy density**
- **Stress tensor** (or momentum fluxes)

- **Angular momentum densities** accessible too:

$$J_i(x) = \epsilon_{ijk}\left(x^j T^{0k}(x) - x^k T^{0j}(x)\right)$$

...basically, from $\mathbf{x} \times \mathbf{p}$

- For physical states, mixture of internal structure & wave packet dependence.
  - Removing wave packet dependence is tricky.
  - Several schemes for dealing with this exist.

Breit frame densities

- **Breit frame** densities most common approach.

$$T^{\mu\nu}_{\mathrm{BF}}(\boldsymbol{x}) \equiv \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{\langle \boldsymbol{q}/2|\hat{T}^{\mu\nu}(\boldsymbol{x})|-\boldsymbol{q}/2\rangle}{2\sqrt{m^2+\boldsymbol{q}^2/4}}$$

  - Original derivation by Sachs erroneous (see Miller, PRC99 (2019) 035202)
  - More recent justification by Lorcé et al., EPJC 79 (2019) 89

- Example: spin-zero energy density and stress tensor

$$\mathcal{E}(\boldsymbol{x}) = m \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{\sqrt{1+\boldsymbol{q}^2/4m^2}} \left\{ A(-\boldsymbol{q}^2) + \frac{\boldsymbol{q}^2}{4m^2}\Big(A(-\boldsymbol{q}^2)+D(-\boldsymbol{q}^2)\Big)\right\} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}$$

$$T^{ij}(\boldsymbol{x}) = m \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{\sqrt{1+\boldsymbol{q}^2/4m^2}} \left(\frac{q^i q^j - \boldsymbol{q}^2\delta^{ij}}{2}\right) D(-\boldsymbol{q}^2) e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}$$

See Polyakov & Schweitzer, Int. J. Mod. Phys. A 33 (2018) for a great review

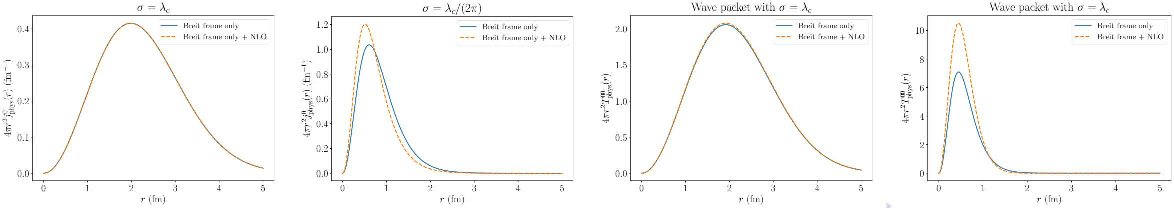
Multipole moment densities

- Consider hadron as a *medium* & wave function as an arbitrary test function
  - Newer idea due to Yang Li et al., PLB 838 (2023) 137676
  - Hadron has potential to contribute to monopole, dipole, etc. densities.
  - Each of these is an intrinsic property!
- Breit frame density emerges as *leading*, **monopole** term in infinite expansion:

$$T^{\mu\nu}(\boldsymbol{x},t) = \int \mathrm{d}^3\boldsymbol{R}\,\mathcal{P}(\boldsymbol{R},t)T^{\mu\nu}_{\mathrm{BF}}(\boldsymbol{x}-\boldsymbol{R}) + \dots$$

  - Higher-order (e.g. quadrupole) densities negligible if packet width $\gtrsim \lambda_C$.

  see AF & Miller, 2210.03807



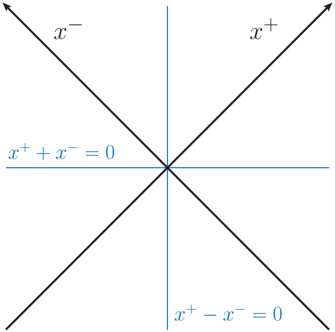
Light front coordinates

- Stark contrast to the non-relativistic case, where:

$$\rho_{\text{phys}}(\mathbf{x}, t) = \int \mathrm{d}^3\mathbf{R} \left|\Psi_{\text{bar}}(\mathbf{R}, t)\right|^2 \rho_{\text{internal}}(\mathbf{x} - \mathbf{R})$$

- The simplicity owes to the **Galilean symmetry** of non-relativistic physics.
- But the Poincaré group has a Galilean subgroup!
- **Light front coordinates** exploit this subgroup to simplify densities.

Light front coordinates



$$x^\pm = \frac{t \pm z}{\sqrt{2}} \qquad \mathbf{x}_\perp = (x, y) \qquad \tau = x^+ = \text{time}$$

Light front: Myths and Facts

- **Myth:** Light front coordinates are a reference frame.
- **Fact:** Light front coordinates can be employed in any reference frame.

- **Myth:** Light front coordinates describe the perspective of light.
- **Fact:** Light front coordinates describe *our* perspective.
  - ...but only in the z direction!
  - Light has no perspective.

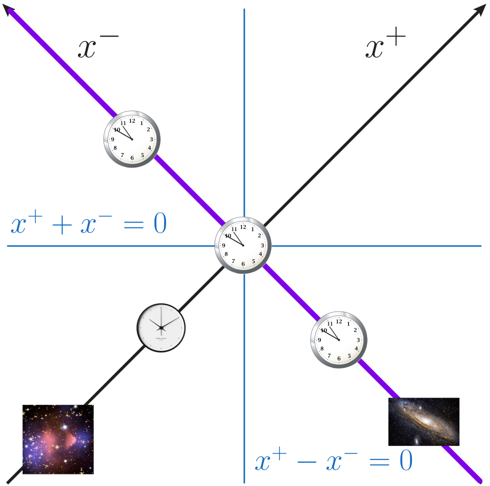
- **Myth:** Light front coordinates come from boosting to infinite momentum.
- **Fact:** Light front coordinates come from redefining:
  1. *Simultaneity*
  2. What we mean by *boosting*
  3. How we break the Poincaré group into generators (*Galilean subgroup*)

Light front & time synchronization

Light front *redefines simultaneity*

- Fixed $x^+ = \frac{t+z}{\sqrt{2}}$ means *simultaneous*
- Look in the $+\hat{z}$ direction...
  - Whatever you **see** *right now*, is **happening** *right now*.
  - *Only* true for $+\hat{z}$ direction though.
- Light front coordinates are what *we* see.
  - ...at least in one fixed direction.
  - (great for small systems—hadrons!)
  - *Not* what light "sees."

More info AF & Miller, arxiv:2302.09171



Terrell rotations

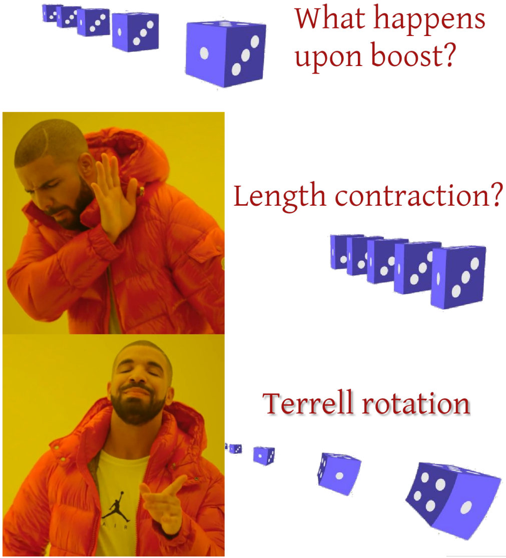
- Lorentz-boosted objects appear *rotated.*
  - **Terrell rotation**
  - Optical effect: contraction + delay
- **Light front transverse boost** *undoes* Terrell rotation:

$$B_x^{(\mathrm{LF})} = \frac{1}{\sqrt{2}}\Big(K_x - J_y\Big)$$

  - Combination of ordinary boost + rotation!
  - Leaves x^+ (time) invariant!
  - Changes p_z, but leaves p^+ invariant:

$$P^+ = \frac{E + p_z}{\sqrt{2}} = \frac{\sqrt{p_z^2 + \mathbf{p}_\perp^2 + M^2} + p_z}{\sqrt{2}}$$

- Dice images by Ute Kraus, `https://www.spacetimetravel.org/`



Galilean subgroup

- Poincaré group has a $(2+1)$D **Galilean subgroup**.
  - x^+ is time and $\mathbf{x}_\perp$ is space under this subgroup.
  - x^- can be integrated out.
  - $P^+ = \frac{1}{\sqrt{2}}(E_\mathbf{p} + p_z)$ is the central charge.
  - x^+ and P^+ are invariant under this subgroup!
- Basically, light front coordinates should give a **fully relativistic** 2D picture that looks like *non-relativistic* physics.
  - But with P^+ in place of M.

$$\frac{\mathrm{d}\mathbf{P}_\perp}{\mathrm{d}x^+} = P^+ \frac{\mathrm{d}^2\mathbf{x}_\perp}{\mathrm{d}{x^+}^2}$$

$$H = P^- = H_{\text{rest}} + \frac{\mathbf{P}_\perp^2}{2P^+}$$

$$\mathbf{v}_\perp = \frac{\mathbf{P}_\perp}{P^+}$$

The cost: lose one spatial dimension (2D densities).

Galilean group and densities

- Physical densities given by expectation values:

$$\rho_{\text{phys}}(x^+, \mathbf{x}_\perp) = \int \mathrm{d}x^- \, \langle \Psi | \hat{\rho}_{\text{op.}}(x) | \Psi \rangle$$

- Galilean subgroup means **barycentric position** and **impact parameter** dependence can be separated.
  - Allows wave packet dependence to be factored out.
  - Instant form coordinates **do not** have this luxury!
- **Simple densities:**

$$\rho_{\text{phys}}(x^+, \mathbf{x}_\perp) = \int \mathrm{d}^2\mathbf{R}_\perp \left| \Psi(\mathbf{R}_\perp, x^+) \right|^2 \rho_{\text{internal}}(\mathbf{x}_\perp - \mathbf{R}_\perp)$$

- **Compound densities:**

$$\rho_{\text{phys}}(x^+, \mathbf{x}_\perp) = \int \mathrm{d}^2\mathbf{R}_\perp \left\{ \mathcal{S}_\Psi^{(1)}(\mathbf{R}_\perp, x^+) \rho_{\text{int.}}^{(1)}(\mathbf{x}_\perp - \mathbf{R}_\perp) + \mathcal{S}_\Psi^{(2)}(\mathbf{R}_\perp, x^+) \rho_{\text{int.}}^{(2)}(\mathbf{x}_\perp - \mathbf{R}_\perp) \right\}$$

Components of the EMT: light front case

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{++}(x) & T^{+1}(x) & T^{+2}(x) & T^{+-}(x) \\ T^{1+}(x) & T^{11}(x) & T^{12}(x) & T^{1-}(x) \\ T^{2+}(x) & T^{21}(x) & T^{22}(x) & T^{2-}(x) \\ T^{-+}(x) & T^{-1}(x) & T^{-2}(x) & T^{--}(x) \end{bmatrix}$$

- **Momentum densities**
- **Energy density**
- **Stress tensor**
- ...and some other things

- **Angular momentum density** (z component) accessible too:

$$J_z(x) = x^1 T^{+2}(x) - x^2 T^{+1}(x)$$

...basically, from $\mathbf{x} \times \mathbf{p}$

- Physical meaning of all 16 components clear in **tilted light front coordinates**.
  - First defined in Blunden, Burkardt, & Miller, PRC**61** (2000) 025026
  - Implications for electromagnetic densities in AF & Miller, arxiv:2302.09171

  (EMT in tilted coordinates is still a work in progress though)

Light front momentum densities

- **Physical** P^+ density is a simple density:

$$T^{++}_{\text{phys}}(\mathbf{x}_\perp) = P^+ \int \mathrm{d}^2\mathbf{R}_\perp \Big|\Psi(\mathbf{R}_\perp, x^+)\Big|^2 T^{++}_{\text{int.}}(\mathbf{x}_\perp - \mathbf{R}_\perp)$$

- Transverse momentum density involves *same internal density*:

$$\mathbf{T}^{+i}_{\text{phys}}(\mathbf{x}_\perp) = \int \mathrm{d}^2\mathbf{R}_\perp \Psi^*(\mathbf{R}_\perp, x^+) \frac{-i\overleftrightarrow{\nabla}^i_\perp}{2} \Psi(\mathbf{R}_\perp, x^+) T^{++}_{\text{int.}}(\mathbf{x}_\perp - \mathbf{R}_\perp)$$

  - But a different smearing function

- Internal density is simple Fourier transform:

$$T^{++}_{\text{int.}}(\mathbf{b}_\perp) = \int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} \frac{\langle p', \mathbf{s}_\perp | \hat{T}^{++}(0) | p, \mathbf{s}_\perp \rangle}{2(P^+)^2} e^{-i\boldsymbol{\Delta}_\perp \cdot \mathbf{b}_\perp}$$

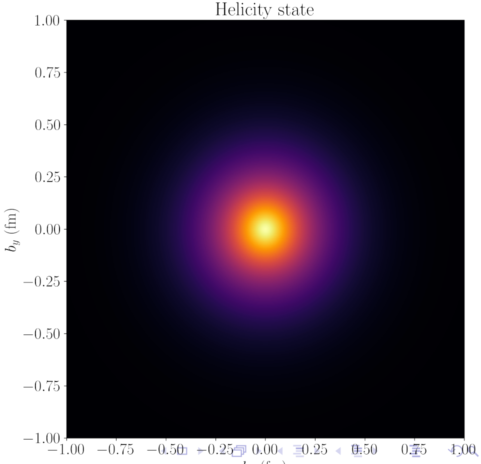
  - Internal structure has polarization dependence.

Light front momentum density

- P^+ density is a 2D Fourier transform:

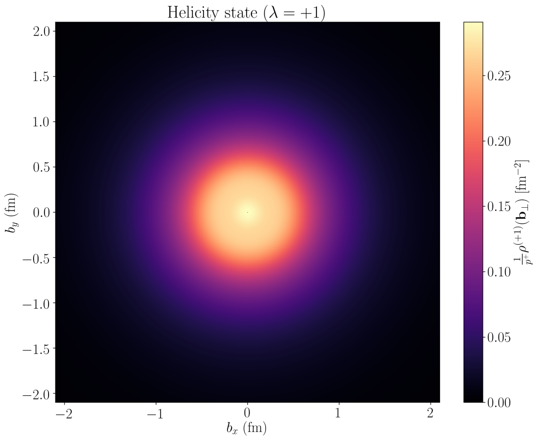
$$\rho^{(\mathrm{LF})}_{P^+}(\mathbf{b}_\perp, \mathbf{s}_\perp) = \int \frac{\mathrm{d}^2\mathbf{\Delta}_\perp}{(2\pi)^2} \frac{\langle p', \mathbf{s}_\perp | T^{++}(0) | p, \mathbf{s}_\perp \rangle}{2(P^+)^2} e^{-i\mathbf{\Delta}_\perp \cdot \mathbf{b}_\perp}$$

  - Works for any polarization state.
  - Structure *relative to* center-of-P^+.
  - Boost invariance: $\mathbf{P}_\perp$ independent!
- Proton dipole model on right.
  - $f_2(1270)$ pole
- Also applicable to massless targets!
  - P^+, not M, is central charge.

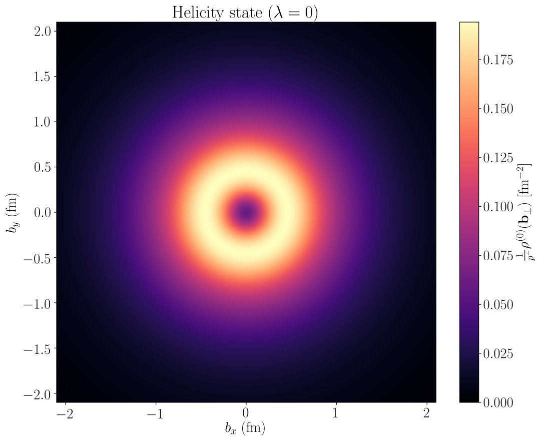


Spin-one targets

Helicity +1



Helicity 0

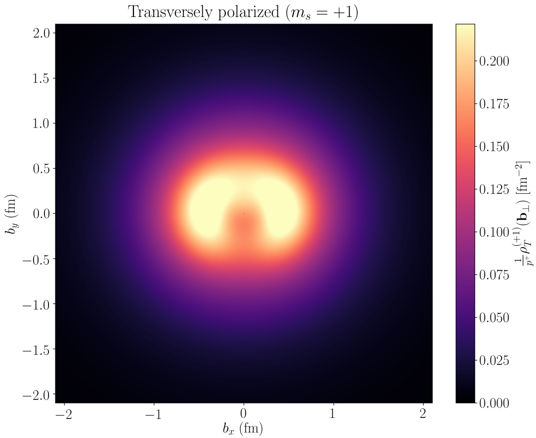


P^+ density depends on helicity for spin-one targets.

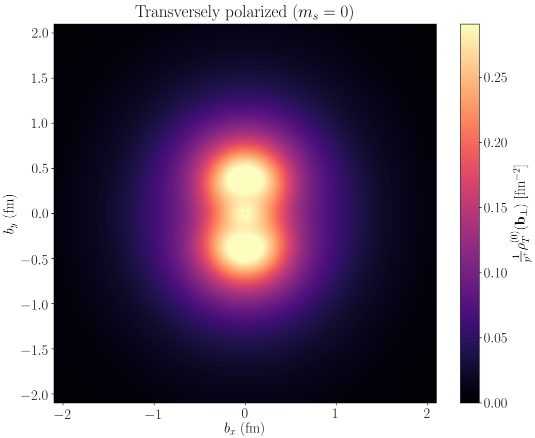
AF & Wim Cosyn, PRD**106** (2022) 114013

Transverse polarization

Transverse, $m_s = +1$



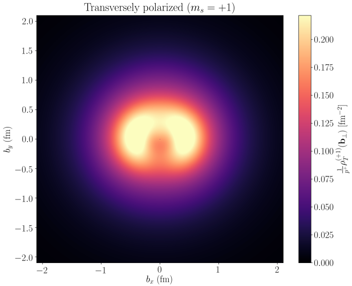
Transverse, $m_s = 0$

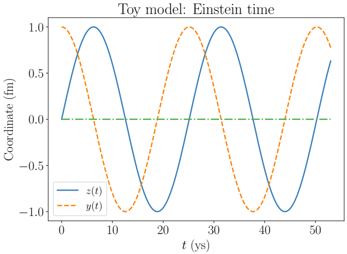


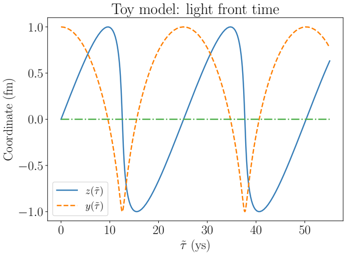
Transverse polarization contains helicity-flip contributions.

AF & Wim Cosyn, PRD**106** (2022) 114013

Why sin ϕ modulations?







- Light takes a finite time to move across deuteron.
- Deuteron spins in the meantime.
- Greater density if spin is against light front.
- Distortion is what *we actually see* (even if stationary!).

More info AF & Miller, arxiv:2302.09171

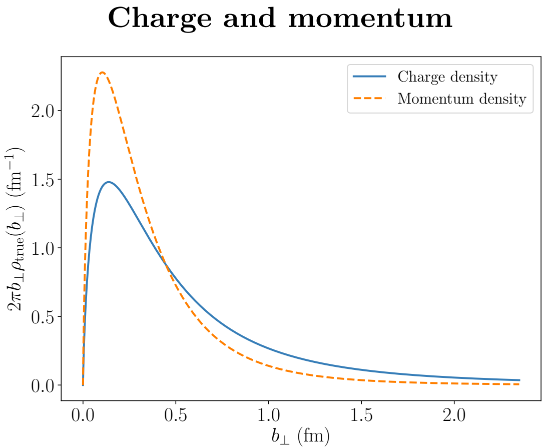
Energy density

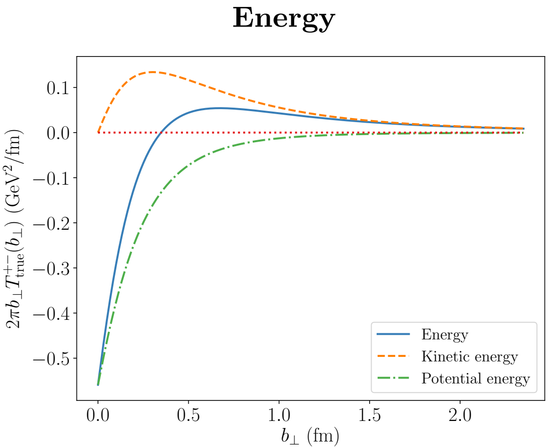
- **Light front energy density** is a *compound density*:

$$T^{+-}_{\text{phys}}(\mathbf{x}_\perp) = \frac{1}{2P^+}\int \mathrm{d}^2\mathbf{R}_\perp \Big\{ \Big|\Psi(\mathbf{R}_\perp, x^+)\Big|^2 T^{+-}_{\text{int.}}(\mathbf{x}_\perp - \mathbf{R}_\perp) - \Psi^*(\mathbf{R}_\perp, x^+)\frac{\overleftrightarrow{\nabla}^2_\perp}{4}\Psi(\mathbf{R}_\perp, x^+) T^{++}_{\text{int.}}(\mathbf{x}_\perp - \mathbf{R}_\perp)\Big\}$$

- First piece is **true intrinsic energy density**
  - Quark mass energy
  - Quark kinetic energy (relative to barycenter)
  - Potential energy & internal stresses
  - Literally the density of the $2P^+P^- - \mathbf{P}_\perp^2$ operator used by light front folks!
- Second piece is **barycentric kinetic energy**
  - It's literally just the $\mathbf{P}_\perp^2/(2P^+)$ density
  - Tells us nothing about internal dynamics
  - Galilean subgroup allows us to isolate it

Example densities from holographic model





Using soft wall holograhpic model of Brodsky & de Teramond, PRD77 (2008) 056007

Stress

- **Stress tensor** is also a *compound density*:

$$T^{ij}_{\text{phys}}(\mathbf{x}_\perp) = \frac{1}{P^+}\int \mathrm{d}^2\mathbf{R}_\perp \left\{ \Big|\Psi(\mathbf{R}_\perp, x^+)\Big|^2 T^{ij}_{\text{int.}}(\mathbf{x}_\perp - \mathbf{R}_\perp) - \Psi^*(\mathbf{R}_\perp, x^+)\frac{\overleftrightarrow{\nabla}^i_\perp \overleftrightarrow{\nabla}^j_\perp}{4}\Psi(\mathbf{R}_\perp, x^+) T^{++}_{\text{int.}}(\mathbf{x}_\perp - \mathbf{R}_\perp) \right\}$$

- First piece is **true intrinsic stress tensor**
  - Stresses seen by comoving observer
  - Static pressures
- Second piece is **stresses from hadron flow**
  - Includes motion of hadron
  - Includes wave function dispersion
- Sum of both gives dynamic pressures

Stress tensor and hadron flow

- Compound form of stress tensor mimics classical continuum mechanics
- In Galilean theory (eg **light front**):

$$T^{ij}(\mathbf{x}, \mathbf{v}, \nabla\mathbf{v}) = \mathbf{v}^i\mathbf{v}^j\rho(\mathbf{x}) + T^{ij}_{\text{pure}}(\mathbf{x}, \nabla\mathbf{v})$$

  - $\mathbf{v}^i\mathbf{v}^j\rho(\mathbf{x})$ depends on wave packet.
- **Comoving stress tensor**:

$$S^{ij}(\mathbf{x}, \nabla\mathbf{v}) = T^{ij}(\mathbf{x}, \mathbf{v} = 0, \nabla\mathbf{v})$$

  - Stresses seen by comoving observer
  - True internal structure of hadron

Example: spin-zero with form factors

- Full stress tensor:

$$\frac{1}{2P^+}\langle p'|\hat{T}^{ij}(0)|p\rangle = P^+ \frac{\mathbf{P}_\perp^i}{P^+}\frac{\mathbf{P}_\perp^j}{P^+} A(t) + \frac{1}{4P^+}\left(\boldsymbol{\Delta}_\perp^i \boldsymbol{\Delta}_\perp^j - \boldsymbol{\Delta}_\perp^2 \delta^{ij}\right) D(t)$$

- **Hadron flow**:

$$V_{\mathrm{LF}}^{ij}(\mathbf{b}_\perp) = \left\langle \frac{\mathbf{P}_\perp^i}{P^+}\frac{\mathbf{P}_\perp^j}{P^+} \right\rangle P^+ \int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} A(t) e^{-i\boldsymbol{\Delta}_\perp\cdot\mathbf{b}_\perp}$$

- **Pure stress tensor**:

$$S_{\mathrm{LF}}^{ij}(\mathbf{b}_\perp) = \frac{1}{4P^+}\int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2}\left(\boldsymbol{\Delta}_\perp^i \boldsymbol{\Delta}_\perp^j - \delta^{ij}\boldsymbol{\Delta}_\perp^2\right) D(t) e^{-i\boldsymbol{\Delta}_\perp\cdot\mathbf{b}_\perp}$$

  - Only **D-term** appears in internal stresses.

Comoving stress tensor, D-term, & intrinsic stresses

- General spins: comoving stress tensor *defines* effective **D-term**.

$$S^{ij}_{\mathrm{LF}}(\mathbf{b}_\perp,\mathbf{s}_\perp) = \frac{1}{4P^+}\int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2}\left(\boldsymbol{\Delta}^i_\perp\boldsymbol{\Delta}^j_\perp - \delta^{ij}\boldsymbol{\Delta}^2_\perp\right) D_{\mathrm{eff}}(\boldsymbol{\Delta}_\perp,\mathbf{s}_\perp)e^{-i\boldsymbol{\Delta}_\perp\cdot\mathbf{b}_\perp}$$

- D-term encodes **intrinsic stresses**, or **comoving stresses**.
  - Stresses *as seen by observer moving with flow* (like a leaf on a river)



Pressures and eigenpressures

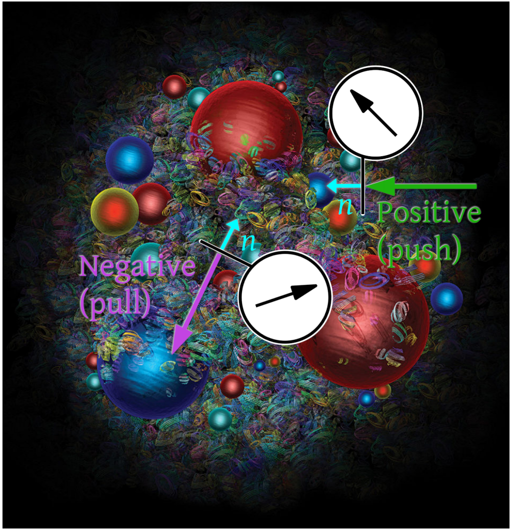
- Pressures from **matrix elements**:

$$p_{\hat{n}}(b_\perp) = \hat{n}_i \hat{n}_j S^{ij}_{\mathrm{LF}}(\mathbf{b}_\perp)$$

  - $\hat{n}$: normal to pressure gauge
- Two **eigenpressures**:

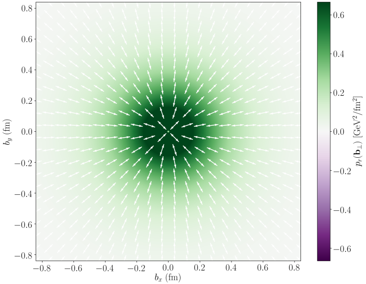
$$S^{ij}_{\mathrm{LF}}(\mathbf{b}_\perp)\hat{n}_j = \lambda_n(b_\perp)\hat{n}_i$$

  - Pressure is isotropic *if and only if* eigenpressures are degenerate!
  - **Anisotropic** pressures in general.
- Meaning of sign?
  - **Positive**: gauge is pushed from both sides.
  - **Negative**: gauge is pulled from both sides.

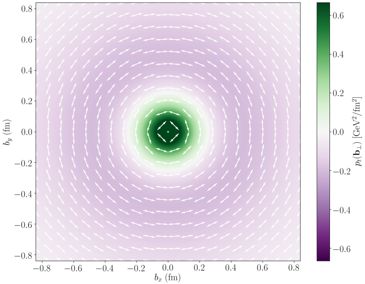


Proton's eigenpressures

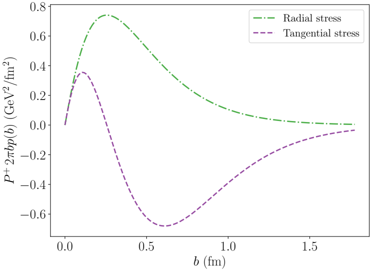
Radial



Tangential



1D reductions



- Helicity proton (spin in $\hat{z}$ direction)
- Tripole model with f_2 and σ poles
- Streses that are eigenvalues of stress tensor
- **Positive radial pressure** related to $D(t) < 0$.
- **Polyakov's conjecture**: $D(0) < 0$ is necessary for stability.

Momentum conservation and force balance

- Conservation law from Noether's theorem:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

- Additional **force balance** equation:

$$\mathbf{F}_\perp^j(\mathbf{x}) = \nabla_{\perp i} S^{ij}(\mathbf{x}) = 0$$

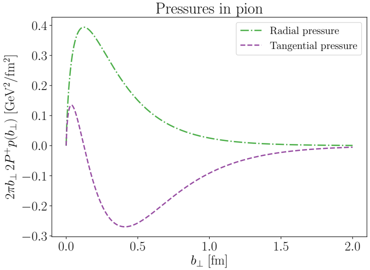
- **Force density acting on a hadron is everywhere zero.**
  - The hadron is in equilibrium.
  - The hadron is not being acted on by outside forces.
  - Pressure plots **are not net force plots**!

Pion pressures and energy density

- Stress tensor and energy density are compound densities.
- Internal (pure) densities given by 2D Fourier transforms:

$$T^{ij}_{\mathrm{pure}}(\mathbf{b}_\perp) = \int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} \frac{\boldsymbol{\Delta}^i_\perp \boldsymbol{\Delta}^j_\perp - \delta^{ij}\boldsymbol{\Delta}^2_\perp}{2} D(-\boldsymbol{\Delta}^2_\perp) e^{-i\boldsymbol{\Delta}_\perp \cdot \mathbf{b}_\perp}$$

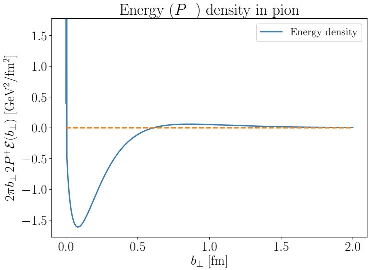
$$\mathcal{E}(\mathbf{b}_\perp) = \int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} \left(m_\pi^2 - \frac{\boldsymbol{\Delta}^2_\perp}{4} \right) A(-\boldsymbol{\Delta}^2_\perp) e^{-i\boldsymbol{\Delta}_\perp \cdot \mathbf{b}_\perp} - \delta_{ij} T^{ij}_{\mathrm{pure}}(\mathbf{b}_\perp)$$



- Phenomenological form factors:

$$A(t) = \frac{1}{1 - t/m^2_{f_2}} \qquad m_{f_2} = 1270 \text{ MeV}$$

$$D(t) = \frac{-1}{(1 - t/m^2_{f_2})(1 - t/m^2_\sigma)} \qquad m_\sigma = 630 \text{ MeV}$$



- Forms inspired by Masjuan *et al* [PRD87 (2013) 014005]
- Poles chosen to match Kumano's radii [PRD97 (2018) 014020]
- AF & Gerald Miller arxiv:2210.03807 for more info!

Simple densities and wave packet localization

- **Simple densities:**

$$\rho_{\text{phys}}(x^+, \mathbf{x}_\perp) = \int \mathrm{d}^2\mathbf{R}_\perp \left|\Psi(\mathbf{R}_\perp, x^+)\right|^2 \rho_{\text{internal}}(\mathbf{x}_\perp - \mathbf{R}_\perp)$$

- Localization at $x^+ = 0$ means:

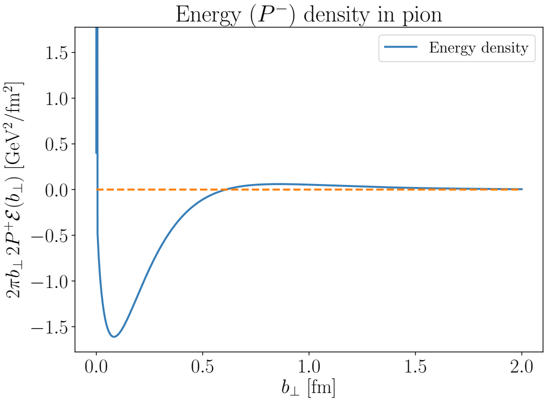
$$\left|\Psi(\mathbf{R}_\perp, 0)\right|^2 \to \delta^{(2)}(\mathbf{R}_\perp)\,, \qquad \rho_{\text{phys}}(0, \mathbf{x}_\perp) \to \rho_{\text{internal}}(\mathbf{x}_\perp)$$

- Common practice has thus been to localize to isolate internal densities.
  - Burkardt & Diehl pioneered this idea for light front densities

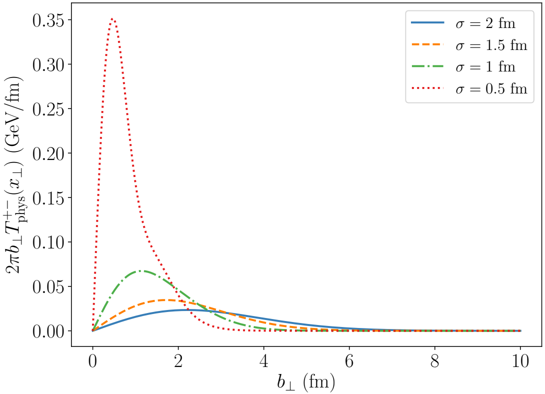
    Int. J. Mod. Phys. A18 (2003) 173, Eur.Phys.J.C 25 (2002) 223-232
  - Works in the cases they considered ...
  - ...but fails for compound densities.

Pion energy and wave packet localization

Internal energy

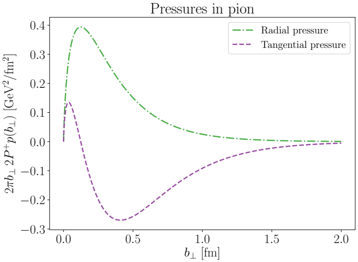


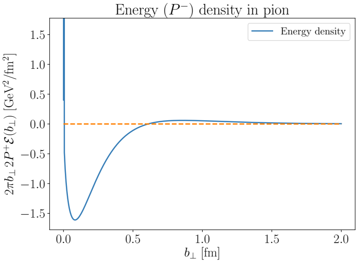
Physical energy (packet dependent)

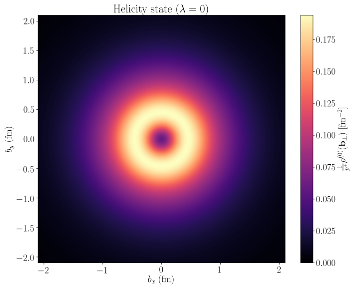


- Energy density is a compound density.
- Localized packets do not give internal density in this case.
- Packet localization only works for simple densities.

Conclusions & outlook







- The **energy-momentum tensor** encodes interesting internal properties of hadrons.
  - Energy, momentum & stress distributions among them.
- There are subtleties in how to identify "internal" properties here.
  - Light front densities & multipole moment densities both promising routes.
  - Studies in tilted light front coordinates remain to be done.

...and, most importantly:

Thanks for your time and attention!