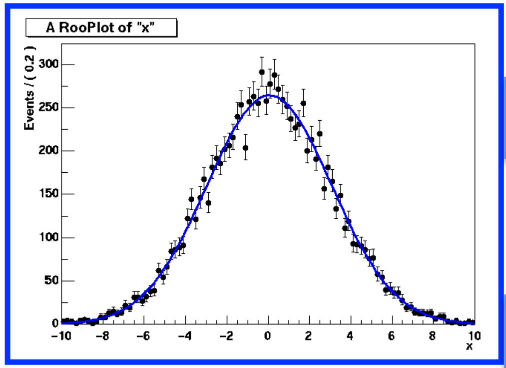
Partial Wave Analysis

2017.07.19

Part 1: basic procedures

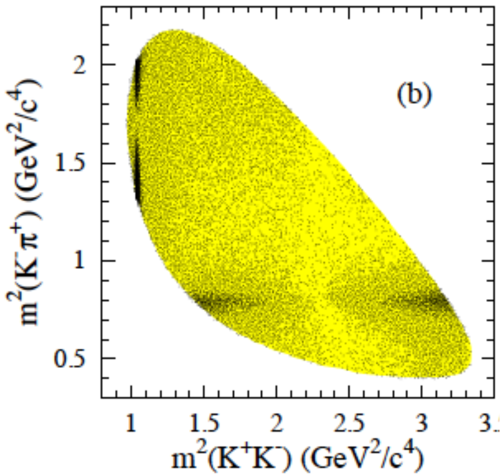
1. **Likelihood construction.**
2. **Fit**
3. **Projection**
4. **Generate DIY mc**

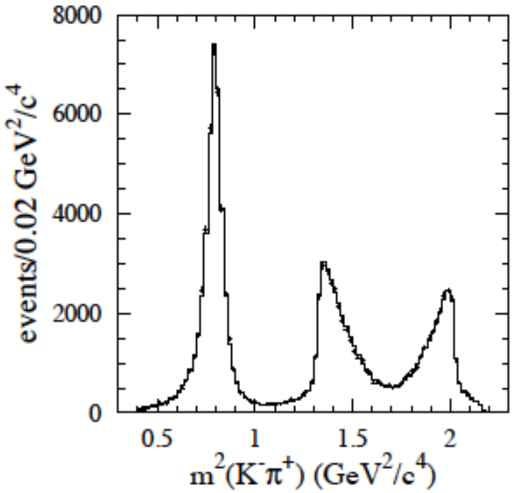
- Partial Wave Analysis(PWA) is just a fit, nothing mysterious.

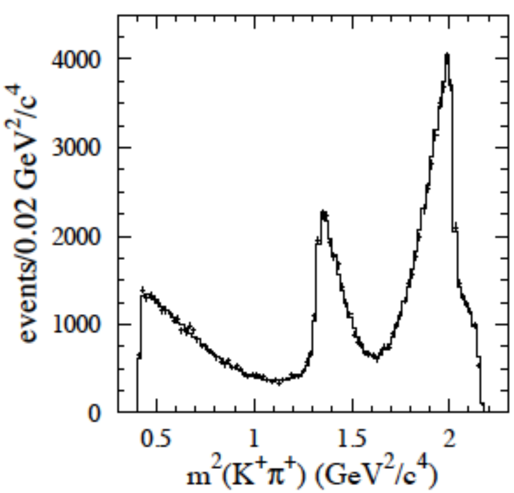


One dimensional simple fit, using one invariant mass as input

Two dimension fit, dalitz plot analysis, using two mass as input







More dimensions, Partial wave analysis, or Amplitude analysis.
Usually using four momentum of final states as input, diffent variables like the invariant Mass, angles can be calculated using the momentum to construct the PDF.

From Li Xiaoling

Construct likelihood function

(four momentum)

ξ is the physical quantity measured by experiment
$\omega(\xi)$ is the probability density to produce it $\omega(\xi) = \frac{d\sigma}{d\Phi_i}$

The differential cross section: $\frac{d\sigma}{d\Phi_i} = |\sum_j A_j|^2$

$\varepsilon(\xi)$ is the efficiency， $P(\xi)$ is the probability to observe it ,
and $P(\xi)$ is defined as: $P(\xi) = \frac{\omega(\xi)\epsilon(\xi)}{\int d\xi\omega(\xi)\epsilon(\xi)}$

For n events, the probability density:

$$P(\xi_i, \xi_2, \cdots, \xi_n) = \prod_{i=1}^{n} P(\xi_i) = \prod_{i=1}^{n} \frac{\omega(\xi_i)\epsilon(\xi_i)}{\int d\xi\omega(\xi)\epsilon(\xi)}$$

$$\ln P(\xi_i, \xi_2, \cdots, \xi_n) = \sum_{i=1}^{n} \ln\left(\frac{\omega(\xi_i)}{\int d\xi\omega(\xi)\epsilon(\xi)}\right) + \underbrace{\sum_{i=1}^{n} \ln\epsilon(\xi_i)}_{constant}$$

Definition of likelihood function: $\mathcal{L} = P(\xi_i, \xi_2, \cdots, \xi_n)$

The logarithm of it: $\ln\mathcal{L} = \ln P(\xi_i, \xi_2, \cdots, \xi_n) = \sum_{i=1}^{n} \ln\left(\frac{d\sigma}{d\Phi_i}/\sigma\right)$

The total observed cross section: $\sigma = \int d\xi\omega(\xi)\epsilon(\xi)$

MC normalization

The total observed cross section is calculated by MC integration:

$$\sigma = \int d\xi \omega(\xi)\epsilon(\xi) = \sum_i \Delta\xi_i \omega(\xi_i)\epsilon(\xi_i)$$
$$= \frac{1}{N_{gen}} \sum_i N_{gen} \Delta\xi_i \omega(\xi_i)\epsilon(\xi_i)$$
$$= \frac{1}{N_{gen}} \sum_i N_{\xi_i} \omega(\xi_i) = \frac{1}{N_{gen}} \sum_{i=1}^{N_{MC}} \omega(\xi_i)$$

The total observed cross section:

$$\sigma = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} \left(\frac{d\sigma}{d\Phi}\right) = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} \left(\sum_j A_j\right)^2$$

The normalization is calculated with MC integration. The integration should go through all possible values, a PHSP MC sample should be generated.
After considering the efficiency, the mc after reconstruction is used to do the MC integration

A trick to reduce the calculating time

$$A_i = \rho_i W_i$$

$$\frac{d\sigma}{d\phi}_k = \sum_i |A_i|^2 = \sum_{i,j} \rho_i \rho_j W_i W_j$$

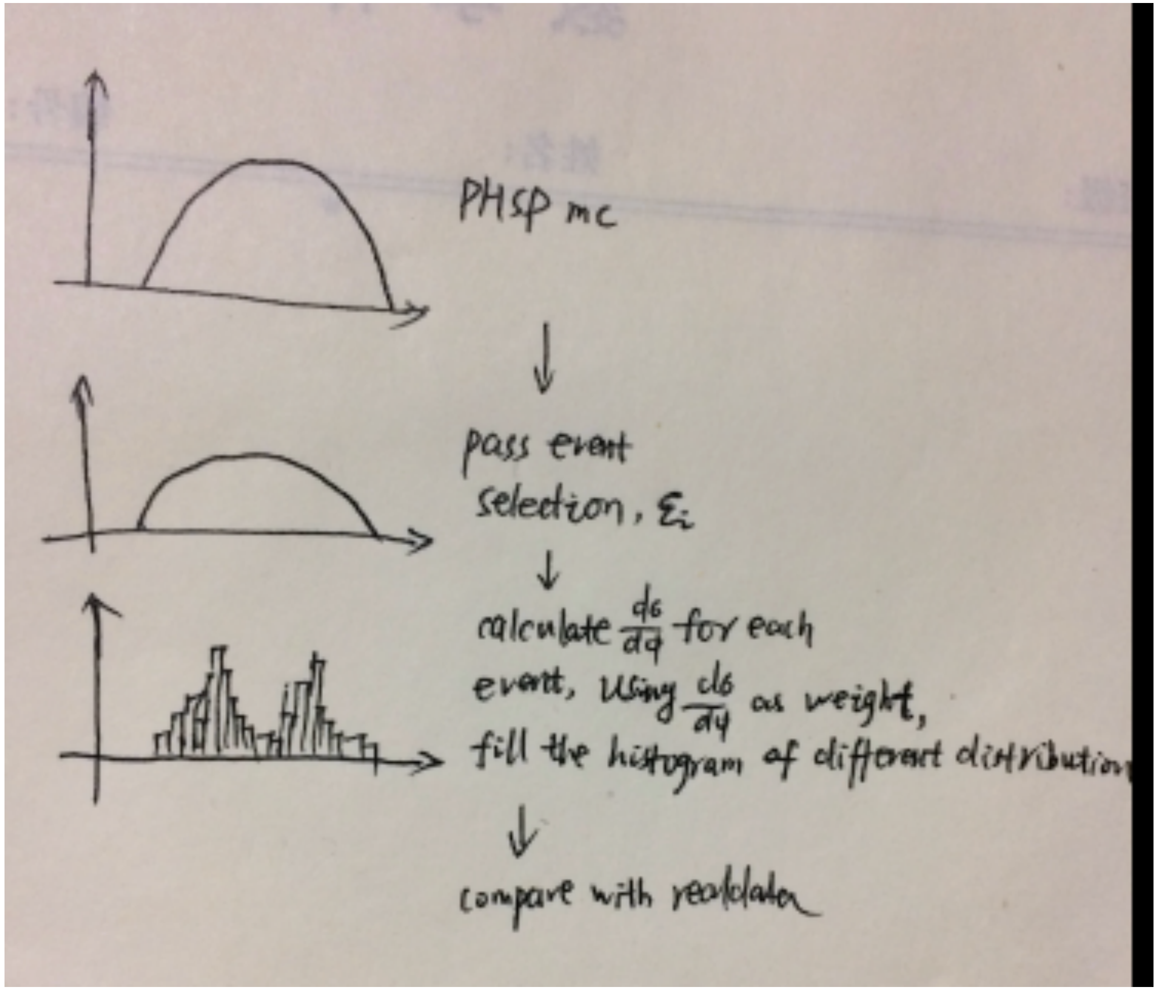
$$\sum_k \frac{d\sigma}{d\phi}_k = \sum_k \sum_i |A_i|^2 = \sum_{i,j} \rho_i \rho_j \boxed{\sum_k W_i W_j}$$

Rho_i is the complex coefficiency, magnitude and phase
Wi is the left part, contains Breit-wigner structure

K means the kth event

If the mass and width of the BW is fixed in the fit, this part can be pre-calculated and saved, so there Is no need to calculate this part every time the coefficiency is changed.

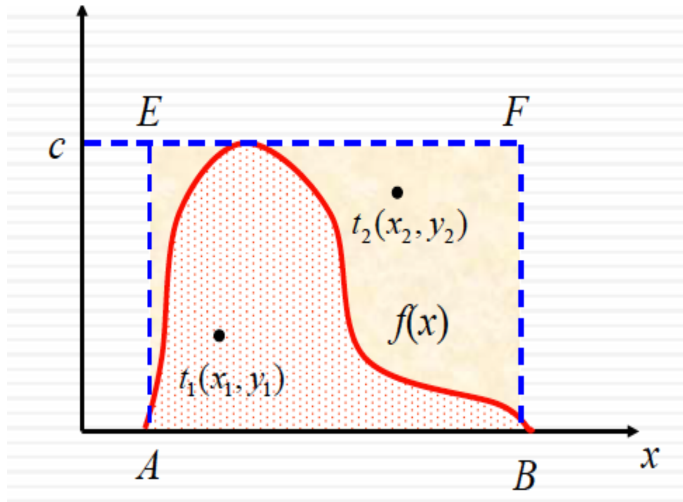
Projection of the fit result and compare with realdata



MC generation with fitted amplitude

- Pick and throw method

1. Generate many phsp event
2. find fmax with the phsp event
3. Generate one phsp event, calculate f(x), then generate a random number rnd In (0,1)
4. if(rnd<f(x)/fmax) pick the event, if(rnd>f(x)/fmax) throw the event.



Part 2:
How to write the amplitue?

Helicity formalism

$$h = \frac{\vec{J}\cdot\vec{P}}{|\vec{P}|} = \vec{J}\cdot\hat{P}$$

Helicity is the projection of spin along the movement direction.
M in |JM> is the projection of spin along the z axis.

$$\begin{aligned}
\mathcal{M}^J_{\lambda_1\lambda_2} &= <\vec{p}\lambda_1; -\vec{p}\lambda_2|\mathcal{M}|JM> \\
&= 4\pi\left(\frac{w}{p}\right)^{\frac{1}{2}} <\theta\phi\lambda_1\lambda_2|JM\lambda_1\lambda_2><JM\lambda_1\lambda_2|\mathcal{M}|JM> \\
&= N_J F^J_{\lambda_1\lambda_2} D^{J*}_{M,\lambda}(\phi,\theta,0), \quad \lambda = \lambda_1 - \lambda_2,
\end{aligned}$$

The cross section(probility) of
a mother particle with spin J
Decay to two pariticle with helicity
Lambda1 and lambda2 along direction
Theta,phi

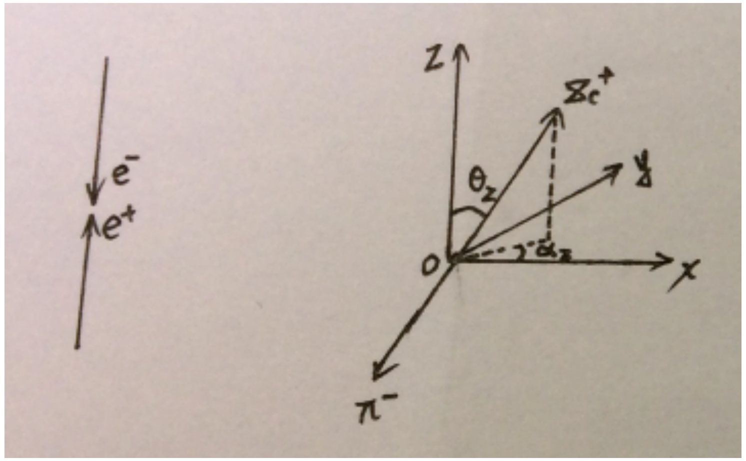
For process a→b+c, The helicity coordinate system of a:
The z axis is along the a's movement direction. (M→λ)
θ,ϕ is the angle of b referring to a's rest frame.

F is called the helicity coupling amplitude. F is rotation invariant.
D is the wigner D function.

the helicity axis for the daughter states. This technique separates out the angular distribution contained in the D function from the problem of finding a proper energy and momentum dependence of the helicity-coupling amplitudes.

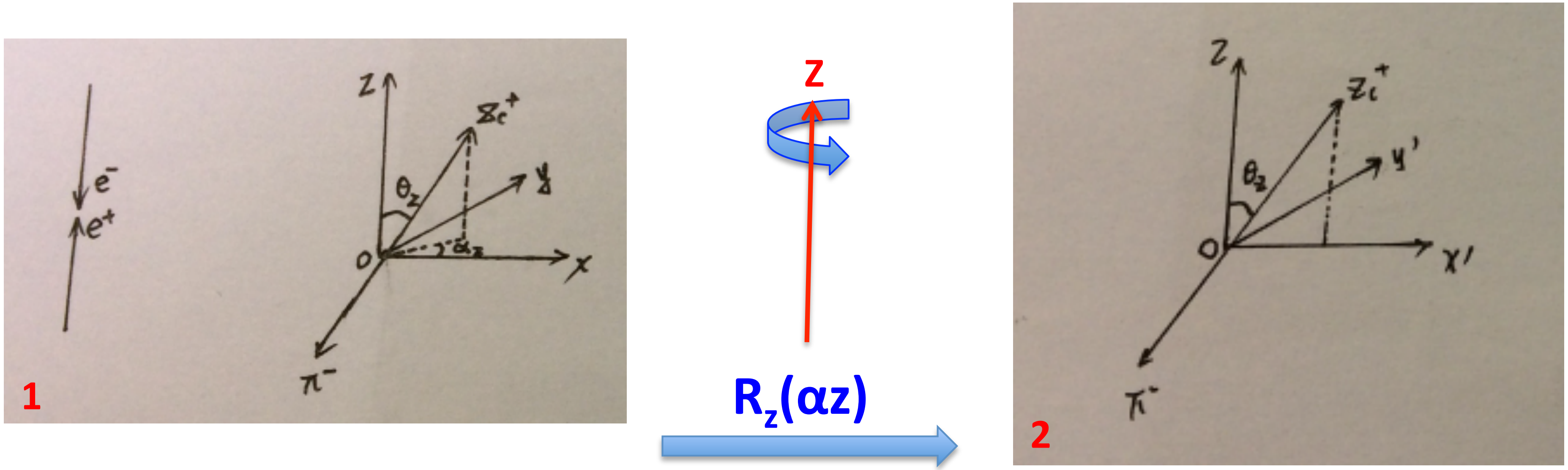
The definition of angles

- The first step $Y \rightarrow Zc^{+} + \pi^{-}$ is defined in lab system, for BESIII, the Z-axis is along the positron beam direction



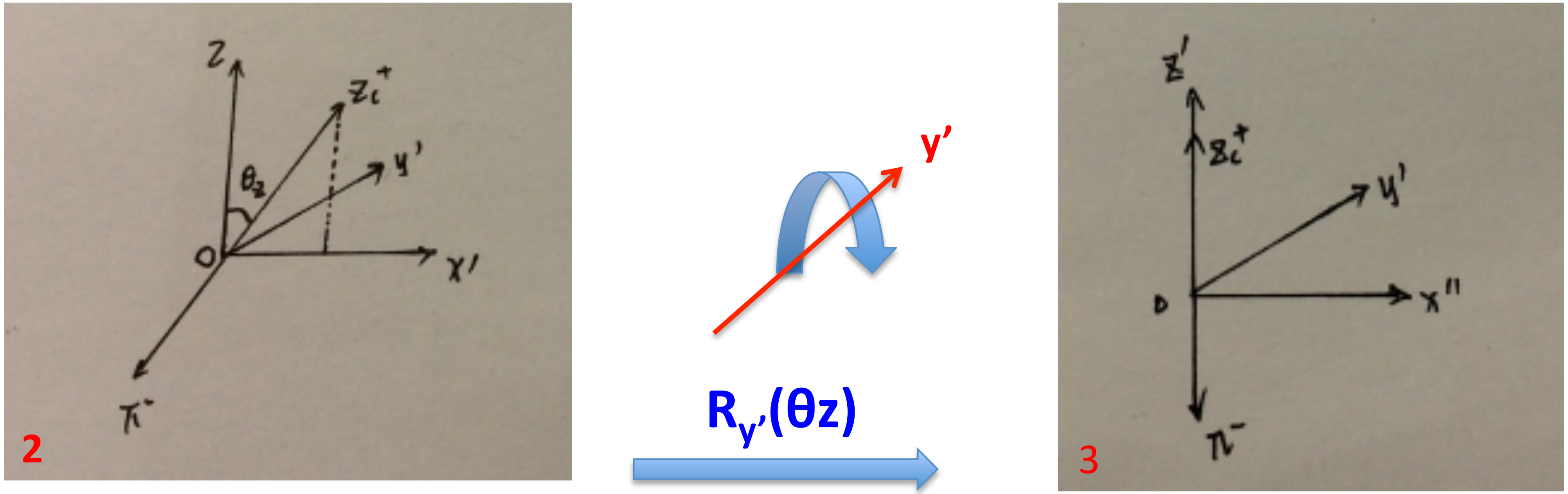
The first step:
$Y \rightarrow Zc^{+} + \pi^{-}$

$$A1 = F^{S_Y}_{\lambda_{Zc}\lambda_{\pi}} D^{S_Y}_{\lambda_Y(\lambda_{Zc}-\lambda_{\pi})}(\alpha_{Zc},\theta_{Zc},0)$$

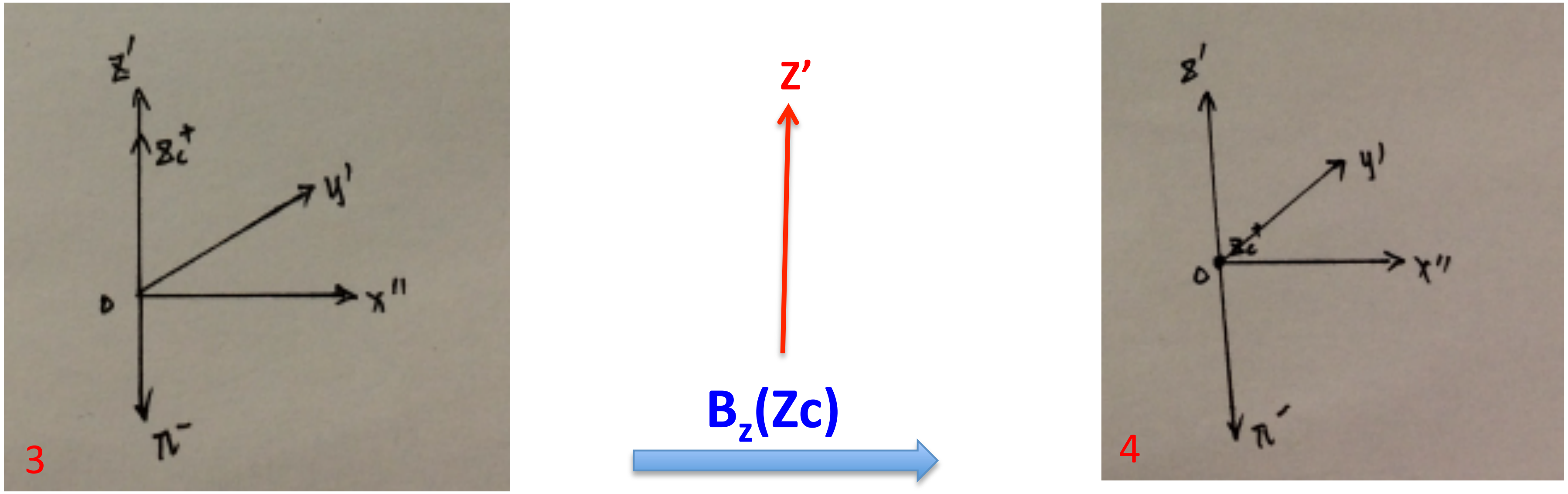


The definition of angles for second step $Zc^+ \rightarrow J/psi + \pi^+$ need to transform to Zc rest frame.

First rotate the frame about Z-axis by angle α_{Zc} , make the Zc^+, π^-, and the new x'-axis in same plane. $y \rightarrow y'$



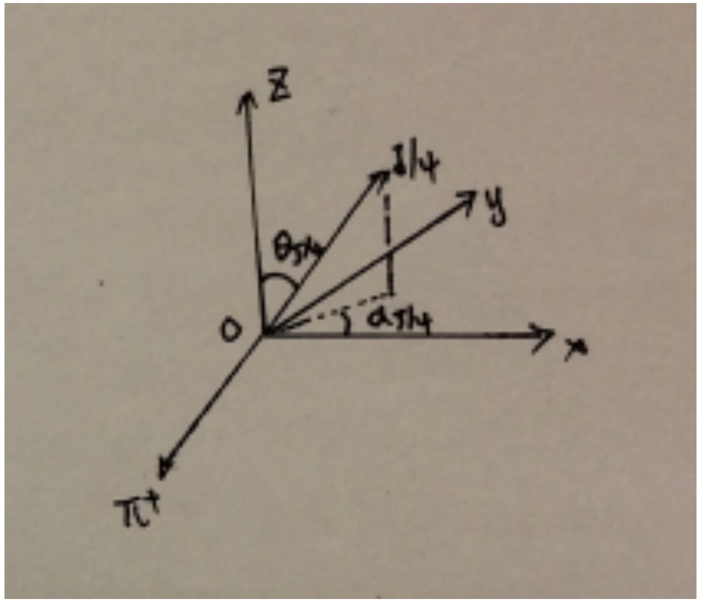
Then rotate the frame about y'-axis by angle θ_{Zc} ,
make the Zc^{+} along the new z'-axis. x'→x''



At last boost Zc along Z'-axis to it's rest frame.

The new frame (x'',y',z') is used to define the angles of next decay step Zc$^+$→J/psi + π^+ .

The angles of the second step



The second step:

Zc$^+$→J/psi + π$^+$

$$A2 = F^{S_{Zc}}_{\lambda_{J/\psi}\lambda_{\pi}} D^{S_{Zc}}_{\lambda_{Zc}(\lambda_{J/\psi}-\lambda_{\pi})}(\alpha_{J/\psi},\theta_{J/\psi},0)$$

For simplicity, replace (x'',y',z') with (x, y, z).

- Each step follow the same procedure.
- If there are other topology such as ee→f_0J/psi, f_0→pipi. then it's another rotation procedure.
- The total amplitude is the direct product of

Each step.

$$A1 = F^{S_Y}_{\lambda_{Zc}\lambda_\pi} D^{S_Y}_{\lambda_Y(\lambda_{Zc}-\lambda_\pi)}(\alpha_{Zc},\theta_{Zc},0)$$

$$A2 = F^{S_{Zc}}_{\lambda_{J/\psi}\lambda_\pi} D^{S_{Zc}}_{\lambda_{Zc}(\lambda_{J/\psi}-\lambda_\pi)}(\alpha_{J/\psi},\theta_{J/\psi},0)$$

$$A3 = F^{S_{J/\psi}}_{\lambda_{l+}\lambda_{l-}} D^{S_{J/\psi}}_{\lambda_{J/\psi}(\lambda_{l+}-\lambda_{l-})}(\alpha_{\mu+},\theta_{\mu+},0)$$

$$A(Zc) = A1\cdot A2\cdot A3 \bullet BW$$

Relation between helicity coupling and LS coupling

$$F^{S_a}_{\lambda_b\lambda_c} = \sum_{l,s}\left(\frac{2l+1}{2S_a+1}\right)^{1/2}(l0s\delta|S_a\delta)(s_a\lambda_a s_b-\lambda_b|s\delta)r^l G^{S_a}_{ls}$$

对称性关系

$$F^J_{\lambda\nu} = \eta_J\eta_s\eta_\sigma(-)^{J-s-\sigma}F^J_{-\lambda-\nu}$$

Example: Take Y→Zc$^+$ + π$^-$, Zc$^+$→J/psi + π$^+$ When Zc is 1+,

1$^-$ → 1$^+$ +0$^-$,

$$F^1_{00} = F^1_{00}$$

$$F^1_{10} = F^1_{-1,0}$$

$$F^1_{1,0} = +g_{01}\sqrt{\frac{1}{3}}r^0 + g_{21}\sqrt{\frac{1}{6}}r^2$$

$$F^1_{0,0} = +g_{01}\sqrt{\frac{1}{3}}r^0 - g_{21}\sqrt{\frac{2}{3}}r^2$$

LS coupling:

$1^- \rightarrow 1^+ + 0^-$,
Total S=1
L=0 or 2.

In the fit, G_{ls} is taken as coefficiency, we can see there are G_{01}, G_{21} two independent Waves, S and D wave.

The independent F function number should be same as independent ls couplings.

For $Zc^+ \rightarrow J/psi + \pi^+$, also have S and D wave.
The total process is (S+D)(S+D)=SS+SD+DS+DD. So there are four independent amplitude.

A simple application: angular distribution calculation with helicity formalism

- $e^+e^- \rightarrow Y(1^{--}) \rightarrow f_0(0^{++}) + h_c(1^{+-}) \rightarrow$ **$\pi^+\pi^- hc$, try to find out the angular distribution of f_0 or hc**

$$F^J_{\lambda\nu} = \eta_J \eta_s \eta_\sigma (-)^{J-s-\sigma} F^J_{-\lambda-\nu}$$

$$F^1_{00} = -F^1_{00} = 0$$

$$F^1_{01} = -F^1_{0,-1}$$

$$\frac{d\sigma}{d\Omega} = \sum_{\lambda Y=\pm 1} \sum_{\lambda h=0,\pm 1} | F^1_{0,\lambda h} D^1_{\lambda Y,-\lambda h} |^2$$

$$= | F^1_{10} |^2 \cdot (| D^1_{1,-1} |^2 + | D^1_{-1,-1} |^2 + | D^1_{11} |^2 + | D^1_{-1,1} |^2)$$

$$\propto \left(\frac{1-\cos\theta}{2}\right)^2 + \left(\frac{1+\cos\theta}{2}\right)^2$$

$$= 1 + \cos^2\theta$$

Coherent sum and incoherent sum

kinematics derived from 4-vectors *decay amplitudes (from theory)*

$$I(\Omega) = \sum_{\alpha} \left| \sum_{\beta} V_{\alpha\beta} A_{\alpha\beta}(\Omega) \right|^2$$

incoherent sum *coherent sum* *production amplitudes (complex fit parameters)*

$e^+e^- \rightarrow Y(1^{--}) \rightarrow X+c \rightarrow$ **a+b+c**

Because Y is from virtual photon, so it's polarized, it helicity only have ± 1, it's helicity values should be incoherently summed.

All the possible helicity of intermediate states should be coherently summed.

The helicity of final states should generally be summed incoherently.

Covariant tensor formalism

基本的构造规则如下：

- 可以使用的成分包括：初末态粒子的自旋波函数（极化矢量）$\phi^*(m)$，纯轨道角动量张量 $\tilde{t}^{(l)}$ 和协变自旋波函数$\omega(m_s)$ 和$\epsilon(m_\sigma)$ 耦合成的总自旋波函数，初态粒子的四动量 p^μ 以及 $\epsilon_{\mu\nu\sigma\gamma}$ 和Lorentz度规 $g_{\mu\nu}$；
- 振幅必须保持Lorentz不变，宇称守恒等对称性的要求；
- 要把符合上面要求的可能振幅按照和 $L-S$ 耦合的概念全部构造出来，个数正好的独立分波数目相同；
- 为了保持宇称守恒，当$(J+s_1+s_2+l)$为奇数时需要$\epsilon_{\mu\nu\sigma\gamma}p^\mu$，其他情况则不需要。

If the final states are spin zero, then there will be only orbital angular momentum function

$$\tilde{t}^{(1)}_{\mu} = P^{(1)}_{\mu\nu} r^\nu = -r_\mu + \frac{m^2-\mu^2}{w^2} p_\mu = \tilde{r}_\mu$$

$$\tilde{t}^{(2)}_{\mu\nu} = P^{(2)}_{\mu\nu\alpha\beta} r^\alpha r^\beta = \tilde{r}_\mu \tilde{r}_\nu - \frac{1}{3}(\tilde{r}\cdot\tilde{r})\tilde{g}_{\mu\nu}(\vec{p})$$

$$\tilde{t}^{(3)}_{\mu\nu\lambda} = \tilde{r}_\mu \tilde{r}_\nu \tilde{r}_\lambda - \frac{1}{5}(\tilde{r}\cdot\tilde{r})[\tilde{g}_{\mu\nu}(\vec{p})\tilde{r}_\lambda + \tilde{g}_{\nu\lambda}(\vec{p})\tilde{r}_\mu + \tilde{g}_{\lambda\mu}(\vec{p})\tilde{r}_\nu]$$

Covariant tensor formalism

Eur. Phys. J. A 16, 537–547 (2003)

For X being a $b_1(1^{+-})$ state, there are four independent amplitudes since both $\psi \to b_1\pi$ and $b_1 \to \phi\pi$ can have both S and D waves:

$$U^{\mu}_{b_1SS} = \tilde{g}^{\mu\nu}_{(123)}\tilde{t}^{(1)}_{(12)\nu}f^{(\phi)}_{(12)}f^{(b_1)}_{(123)} + \tilde{g}^{\mu\nu}_{(124)}\tilde{t}^{(1)}_{(12)\nu}f^{(\phi)}_{(12)}f^{(b_1)}_{(124)}, \tag{47}$$

$$U^{\mu}_{b_1SD} = \tilde{t}^{(2)\mu\nu}_{(\phi 3)}\tilde{t}^{(1)}_{(12)\nu}f^{(\phi)}_{(12)}f^{(b_1)}_{(123)} + \tilde{t}^{(2)\mu\nu}_{(\phi 4)}\tilde{t}^{(1)}_{(12)\nu}f^{(\phi)}_{(12)}f^{(b_1)}_{(124)}, \tag{48}$$

$$U^{\mu}_{b_1DS} = \tilde{T}^{(2)\mu\lambda}_{(b_1 4)}\tilde{g}_{(123)\lambda\nu}\tilde{t}^{(1)\nu}_{(12)}f^{(\phi)}_{(12)}f^{(b_1)}_{(123)} + \tilde{T}^{(2)\mu\lambda}_{(b_1 3)}\tilde{g}_{(124)\lambda\nu}\tilde{t}^{(1)\nu}_{(12)}f^{(\phi)}_{(12)}f^{(b_1)}_{(124)}, \tag{49}$$

$$U^{\mu}_{b_1DD} = \tilde{T}^{(2)\mu\lambda}_{(b_1 4)}\tilde{t}^{(2)}_{(\phi 3)\lambda\nu}\tilde{t}^{(1)\nu}_{(12)}f^{(\phi)}_{(12)}f^{(b_1)}_{(123)} + \tilde{T}^{(2)\mu\lambda}_{(b_1 3)}\tilde{t}^{(2)}_{(\phi 4)\lambda\nu}\tilde{t}^{(1)\nu}_{(12)}f^{(\phi)}_{(12)}f^{(b_1)}_{(124)}. \tag{50}$$

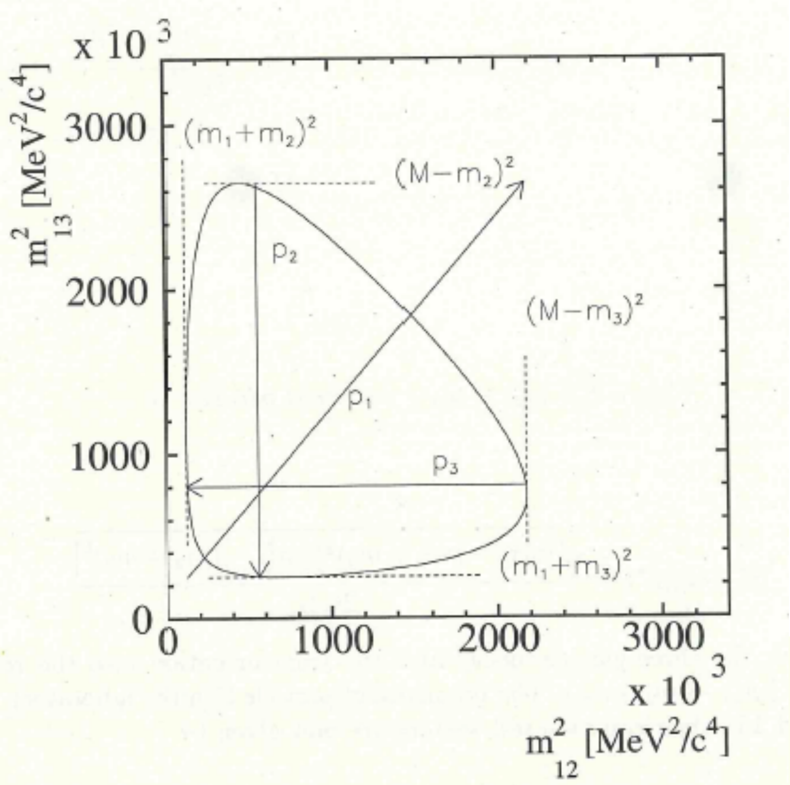
Dalitz Plot analysis

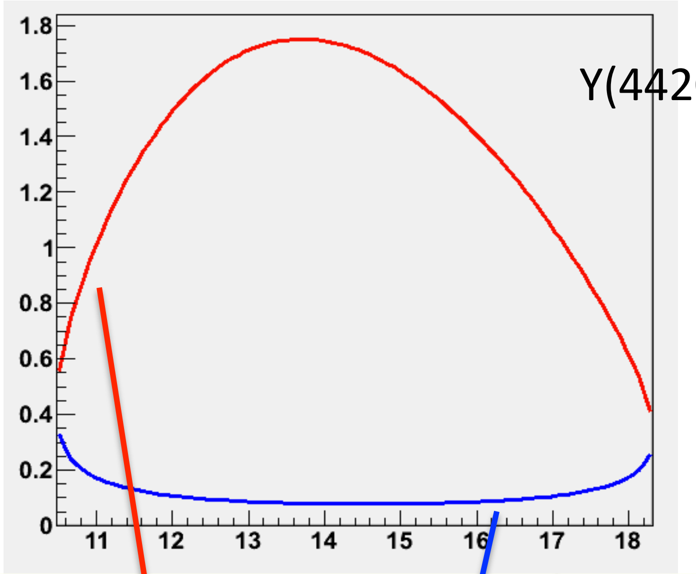
a→ b+ c +d, all the particle should be spin 0

表 6.1: Dalitz-Plot 分析中的独立变量数目。

末态3个四矢量	12	
四动量守恒限制	−4	
三个质壳条件	−3	3个末态的不变质量
空间各向同性 (无自旋假设)	−3	3个方向的角动量守恒
总和	2	

Contour of dalitz plot





Y(4420)→ π π J/ ψ

Maximum and minimum of $m^2_{13} = m^2_{31}$ are calculated to

$$(m^2_{13})_{\max} = (E^*_1 + E^*_3)^2 - \left(\sqrt{E^{*2}_1 - m^2_1} - \sqrt{E^{*2}_3 - m^2_3}\right)^2$$

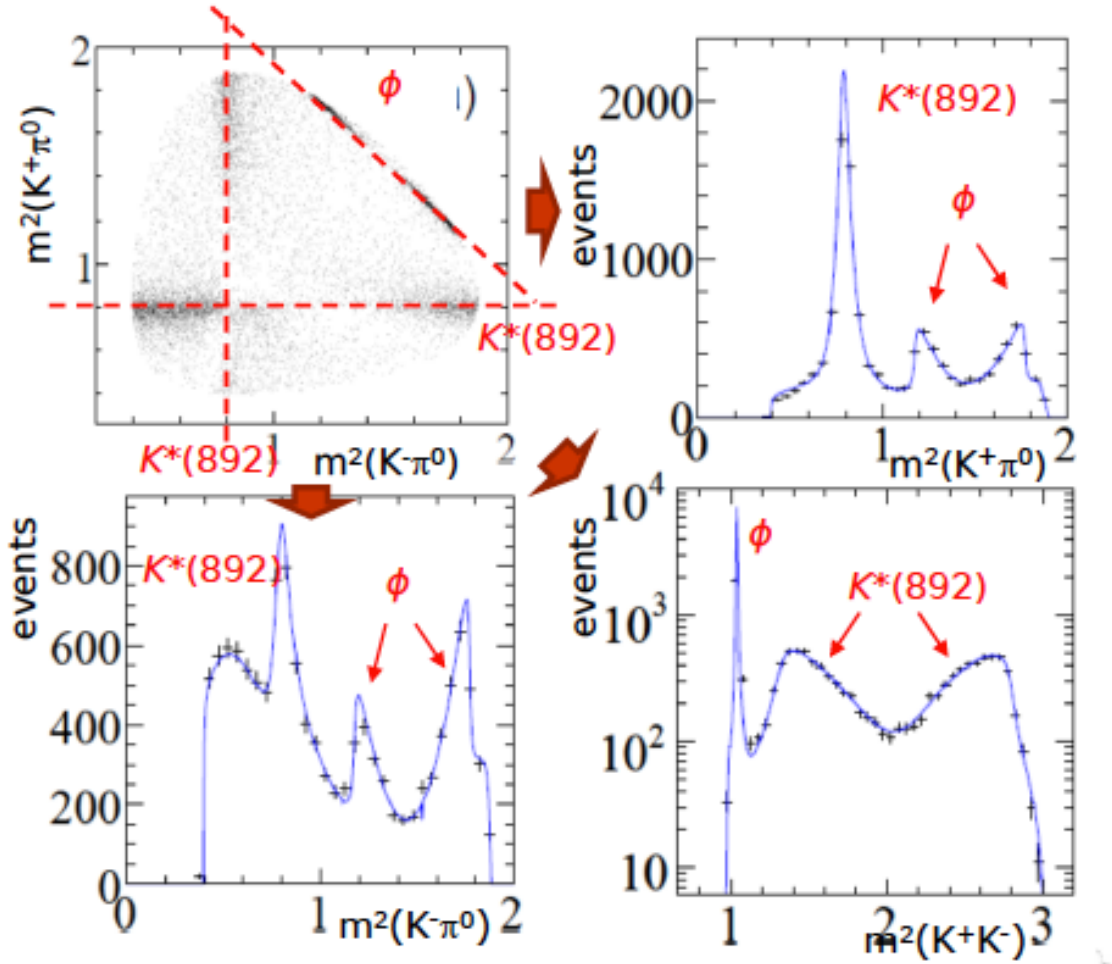
$$(m^2_{13})_{\min} = (E^*_1 + E^*_3)^2 - \left(\sqrt{E^{*2}_1 - m^2_1} + \sqrt{E^{*2}_3 - m^2_3}\right)^2$$

We can calculate the particles energies in the various two-particle rest frames. In the m_{12} rest frame we find

$$E^*_1 = \frac{m^2_{12} - m^2_2 + m^2_1}{2m_{12}}, \quad E^*_2 = \frac{m^2_{12} - m^2_1 + m^2_2}{2m_{12}} \quad \text{and} \quad E^*_3 = \frac{M^2 - m^2_{12} - m^2_3}{2m_{12}} \tag{2.28}$$

How to read the dalitz plot

$D_s^+ \to K^+K^-\pi^0$



振幅公式

$$\mathcal{M} = \sum_R c_R \times \mathcal{W}_R \times \Omega_R \times \mathcal{F}_D^L \times \mathcal{F}_R^L,$$

CR: 拟合系数

Wr: Breit-Wigner

Ωr: 角分布

F^L_D: 衰减因子

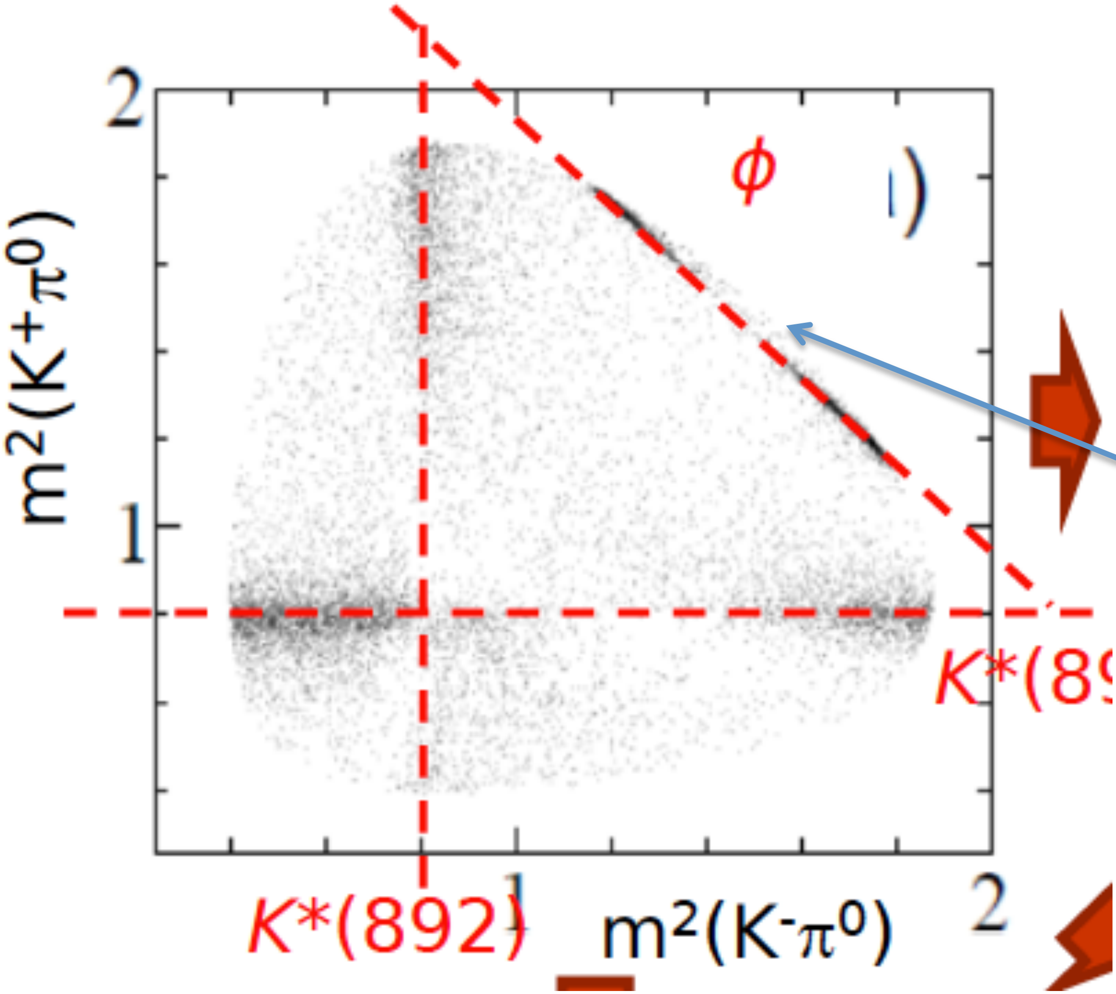
$$\begin{aligned}
\Omega_R^{L=0} &= 1, \\
\Omega_R^{L=1} &= m_{bc}^2 - m_{ac}^2 + \frac{(m_d^2 - m_c^2)(m_a^2 - m_b^2)}{m_{ab}^2}, \\
\Omega_R^{L=2} &= [\Omega_R^{L=1}]^2 - \frac{1}{3}\left(m_{ab}^2 - 2m_d^2 - 2m_c^2 + \frac{(m_d^2 - m_c^2)^2}{m_{ab}^2}\right)\left(m_{ab}^2 - 2m_a^2 - 2m_b^2 + \frac{(m_a^2 - m_b^2)^2}{m_{ab}^2}\right),
\end{aligned} \tag{9}$$

$D_s^+ \rightarrow K^+ K^- \pi^0$

a b c

For Ds→ φ π , L=1, and Ma=Mb, so

$$\Omega_R^{L=1} = m_{bc}^2 - m_{ac}^2 .$$



$$\Omega_R^{L=1} = m_{bc}^2 - m_{ac}^2$$

$$= m_{K^+\pi^0}^2 - m_{K^-\pi^0}^2$$

=0 when m(k-pi0)=m(K+pi0)
So the ϕ band is divided into two section.