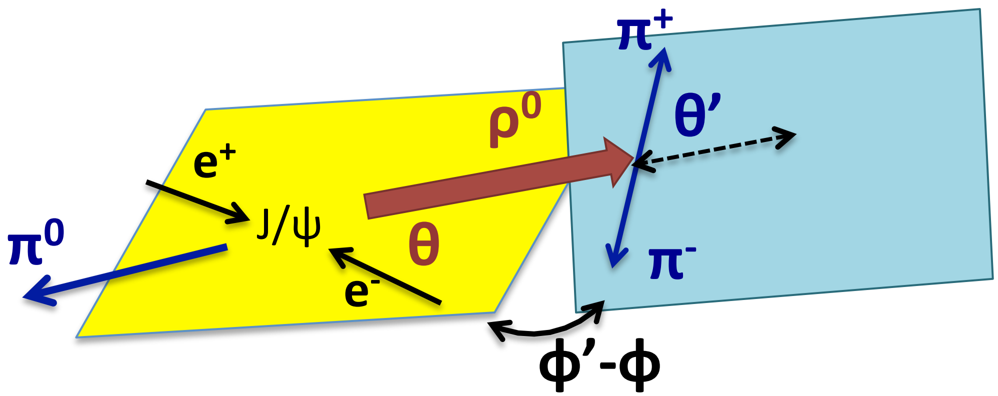
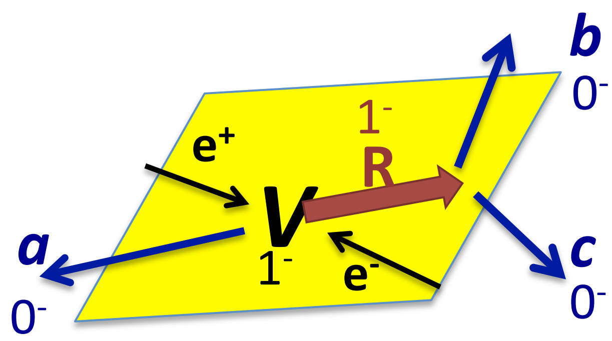
Beyond Dalitz plots



Stephen Lars Olsen

Specifying 3-body final states



$$e^+e^- \to \vec{V} \to a + \vec{R}$$
$$\qquad\qquad\qquad\qquad \hookrightarrow b + c$$

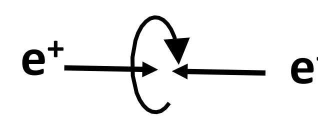
total number of kinematic quantities = 3×3-vectors $=9$ $\quad (p_a^x p_a^y p_a^z),(p_b^x p_b^y \ldots$

energy momentum constraints $=\color{red}{4}$

$$E_a + E_b + E_c = m_V$$
$$\vec{p}_a + \vec{p}_b + \vec{p}_c = 0$$

number of symmetries:

since V is spin=1 $\Rightarrow \color{red}{1}$

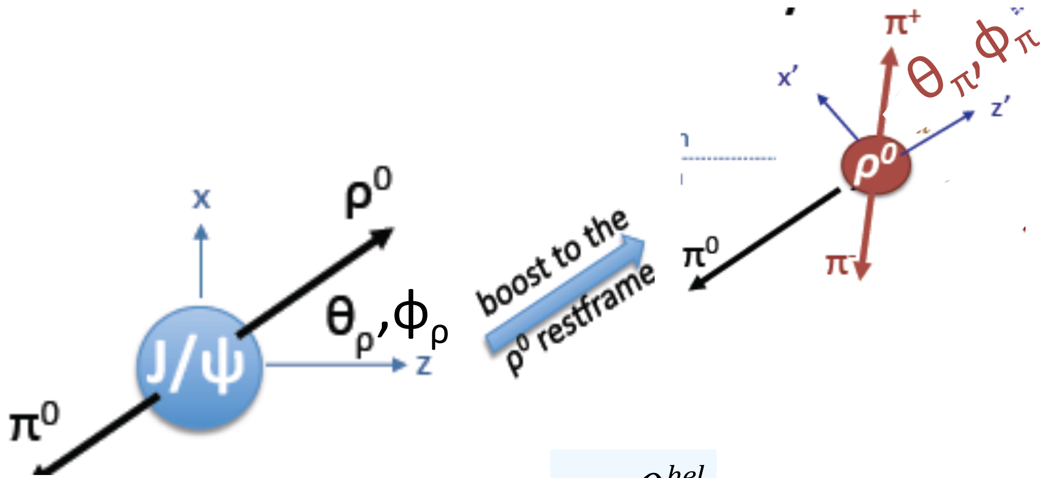


ϕ symmetry

\# of independent variables =4

normal Dalitz plot has only 2 dimensions, not enough

What 4 variables for J/ψ→ρ⁰π⁰?



Dalitz plot variables

$m_{\pi^+\pi^-}$

$\theta_\rho \quad \propto 1+\cos^2\theta_\rho$

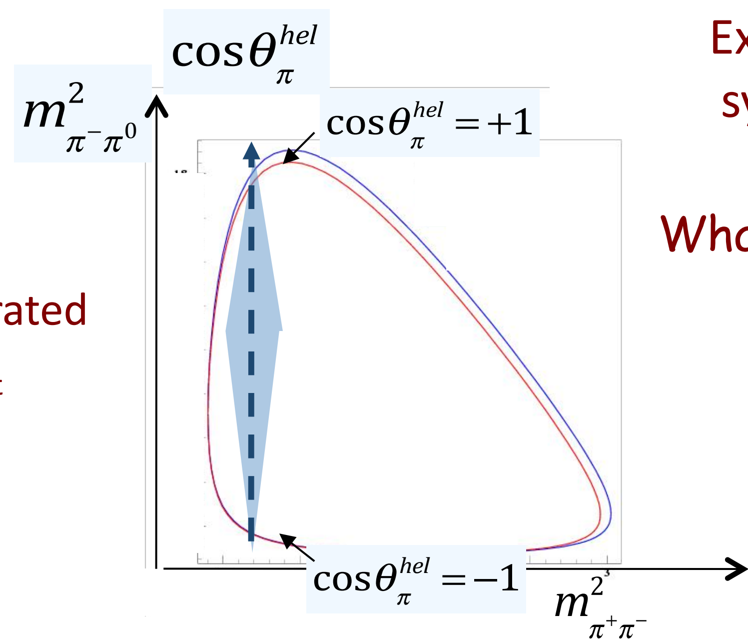
θ_π^{hel}

Excluded by symmetry?

ϕ_ρ

ϕ_π

What is the 4th variable?

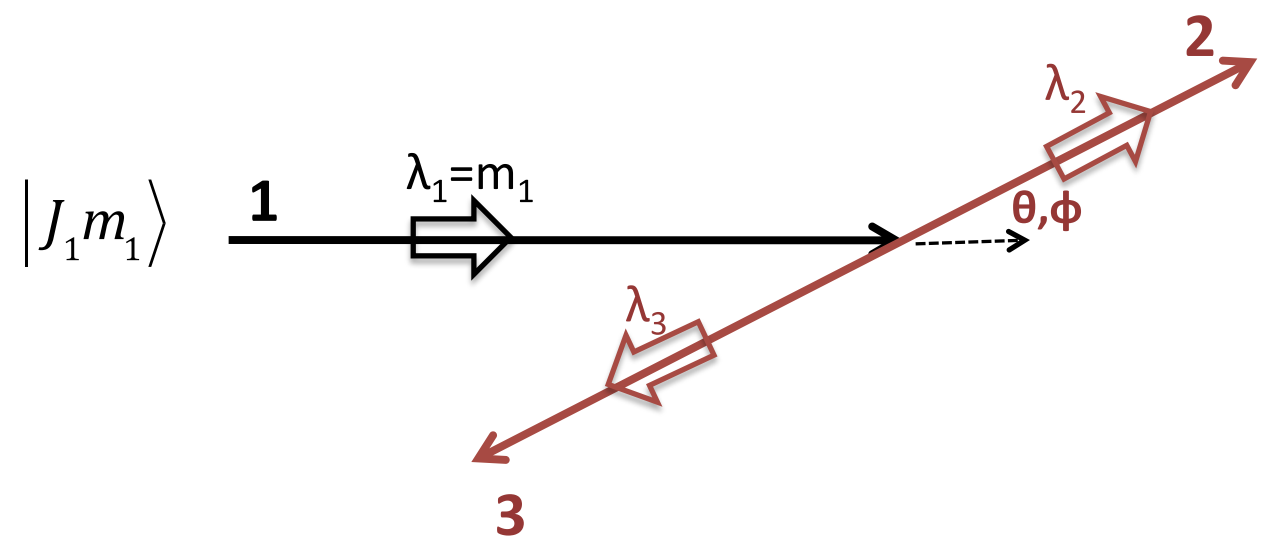


here we have integrated over θ_ρ, ϕ_ρ and ϕ_π

let's do a "full" analysis of $J/\psi \rightarrow \rho^0 \pi^0$

-- do not integrate over any variables --

1→2+3 in the helicity basis



"helicity amplitude" "Wigner D function"

$$A_H(\theta,\phi;m_1,\lambda_2,\lambda_3)=\sqrt{\frac{2J_1+1}{4\pi}}A_{\lambda_2\lambda_3}D^{s_1\,*}_{m_1\lambda_2-\lambda_3}(\theta,\phi)$$

What are the Wigner *D* functions?

simplest version: $D^{j}_{m',m}(\theta,\phi) = e^{-i(m'-m)\phi} d^{j}_{m',m}(\theta)$

$d^{j}_{m'm}(\theta)$ are listed in PDG's Clebsch-Gordon coefficient tables

$$d^{1}_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\ell}_{m,0} = \sqrt{\tfrac{4\pi}{2\ell+1}}\, Y^{m}_{\ell} e^{-im\phi}$$

since $Y^{-m}_{\ell} = (-1)^m Y^{m*}_{\ell} \Rightarrow d^{\ell}_{-m,0} = (-1)^m d^{\ell}_{m,0}$

this means that $d^{1}_{-1,0}(\theta) = -d^{1}_{1,0}(\theta) = +\dfrac{\sin\theta}{\sqrt{2}}$

helicity amplitude ($A_{\lambda_1\lambda_2}$) properties

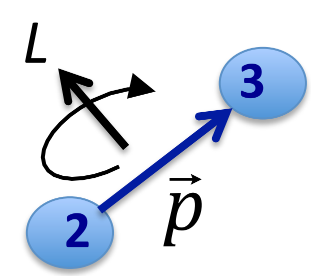
if Parity is conserved in 1→2 + 3: $A_{m'm} = \eta_1\eta_2\eta_3(-1)^{s_1-s_2-s_3} A_{-m'-m}$

true for J/ψ decays, but not for *D* decays $\eta_i = \text{Parity of } i$

Sometimes helicity amplitudes are assumed

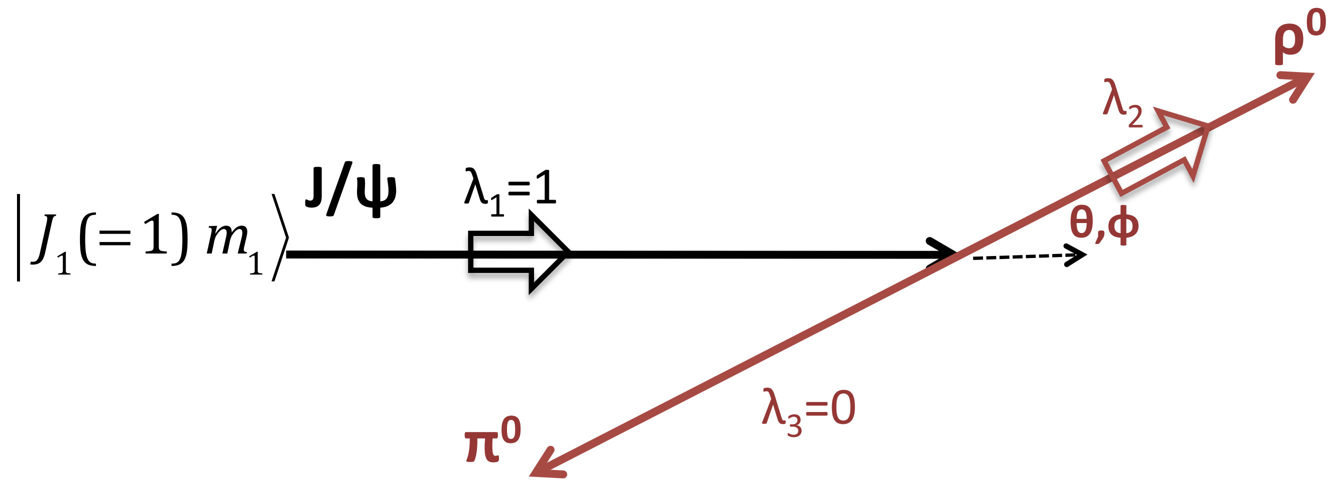
to behave as $A_{\lambda_2\lambda_3} \propto |\vec{p}|^L$

For simplicity, we'll treat them as constants



Let’s apply this to a simple example

$J/\psi \rightarrow \rho^0\pi^0$ in the helicity basis



$$\lambda_2 = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

from slide 4

$$D^j_{m'm}(\theta,\phi) = e^{-i(m'-m)\phi} d^j_{m'm}(\theta)$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$A_H(\theta,\phi;1,\lambda_2,0) = \sqrt{\frac{3}{4\pi}} A_{\lambda_2 0} D^{1*}_{1\lambda_2-0}(\theta,\phi)$$

$$= \sqrt{\frac{3}{4\pi}} \left\{ \begin{array}{c} d^1_{11}(\theta)A_{10} \\ e^{i\phi} d^1_{10}(\theta)A_{00} \\ e^{2i\phi} d^1_{1-1}(\theta)A_{-10} \end{array} \right\} = e^{i\phi}\sqrt{\frac{3}{4\pi}} \left\{ \begin{array}{c} A_{10}e^{-i\phi}(1+\cos\theta)/2 \\ -A_{00}(\sin\theta)/\sqrt{2} \\ A_{-10}e^{i\phi}(1-\cos\theta)/2 \end{array} \right\}$$

Parity is conserved in J/ψ→ρπ

$$\boxed{A_{m'm} = \eta_{J/\psi}\eta_\rho\eta_\pi(-1)^{S_{J/\psi}-s_\rho-s_\pi} A_{-m'-m}}$$

$$A_{10} = (-1)_{J/\psi}(-1)_\rho(-1)_\pi(-1)^{1-1-0} A_{-10}$$
$$\Rightarrow A_{10} = -A_{-10}$$

$$A_{00} = (-1)_{J/\psi}(-1)_\rho(-1)_\pi(-1)^{1-1-0} A_{00}$$
$$\Rightarrow A_{00} = -A_{00} \Rightarrow A_{00} = 0$$

$$A_H(\theta,\phi;1,\lambda_2,0) = \sqrt{\frac{3}{4\pi}} A_{\lambda_2 0} D^{1*}_{1\lambda_2-0}(\theta,\phi)$$
$$= \sqrt{\frac{3}{4\pi}}\left\{\begin{matrix} d^1_{11}(\theta)A_{10} \\ e^{i\phi}d^1_{10}(\theta)A_{00} \\ e^{2i\phi}d^1_{1-1}(\theta)A_{-10}\end{matrix}\right\} = e^{i\phi}\sqrt{\frac{3}{4\pi}}\left\{\begin{matrix} A_{10}e^{-i\phi}(1+\cos\theta)/2 \\ -A_{00}(\sin\theta)/\sqrt{2} \\ A_{-10}e^{i\phi}(1-\cos\theta)/2\end{matrix}\right\} = A_{10}e^{i\phi}\sqrt{\frac{3}{4\pi}}\left\{\begin{matrix} e^{-i\phi}(1+\cos\theta)/2 \\ 0 \\ -e^{i\phi}(1-\cos\theta)/2\end{matrix}\right\}$$

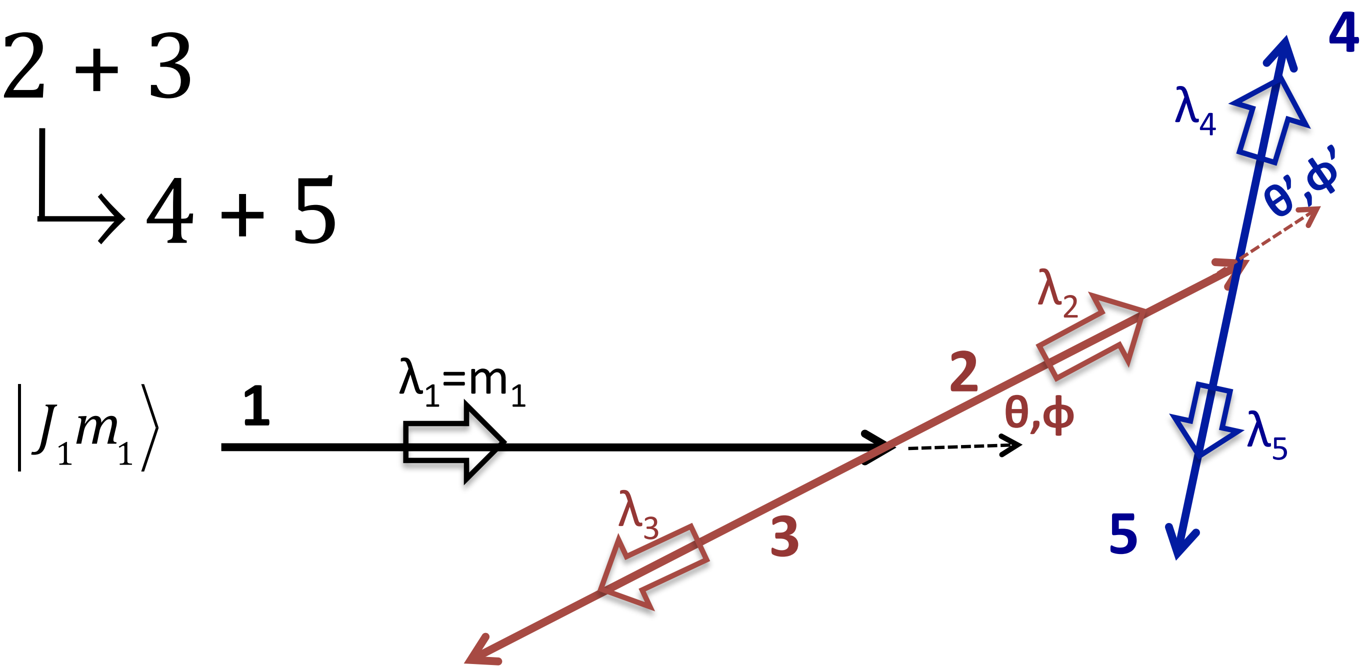
$$\frac{d\Gamma}{d(\theta,\varphi,1,1,0)} = \frac{3|A_{10}|^2}{4\pi}\left|e^{-i\phi}(1+\cos\theta)/2\right|^2 = \frac{3|A_{10}|^2}{16\pi}(1+2\cos\theta+\cos^2\theta)$$

incoherent

$$\frac{d\Gamma}{d(\theta,\varphi,1,-1,0)} = \frac{3|A_{10}|^2}{4\pi}\left|e^{i\phi}(1-\cos\theta)/2\right|^2 = \frac{3|A_{10}|^2}{16\pi}(1-2\cos\theta+\cos^2\theta)$$

add another decay

$$1 \rightarrow 2 + 3$$
$$\quad \hookrightarrow 4 + 5$$



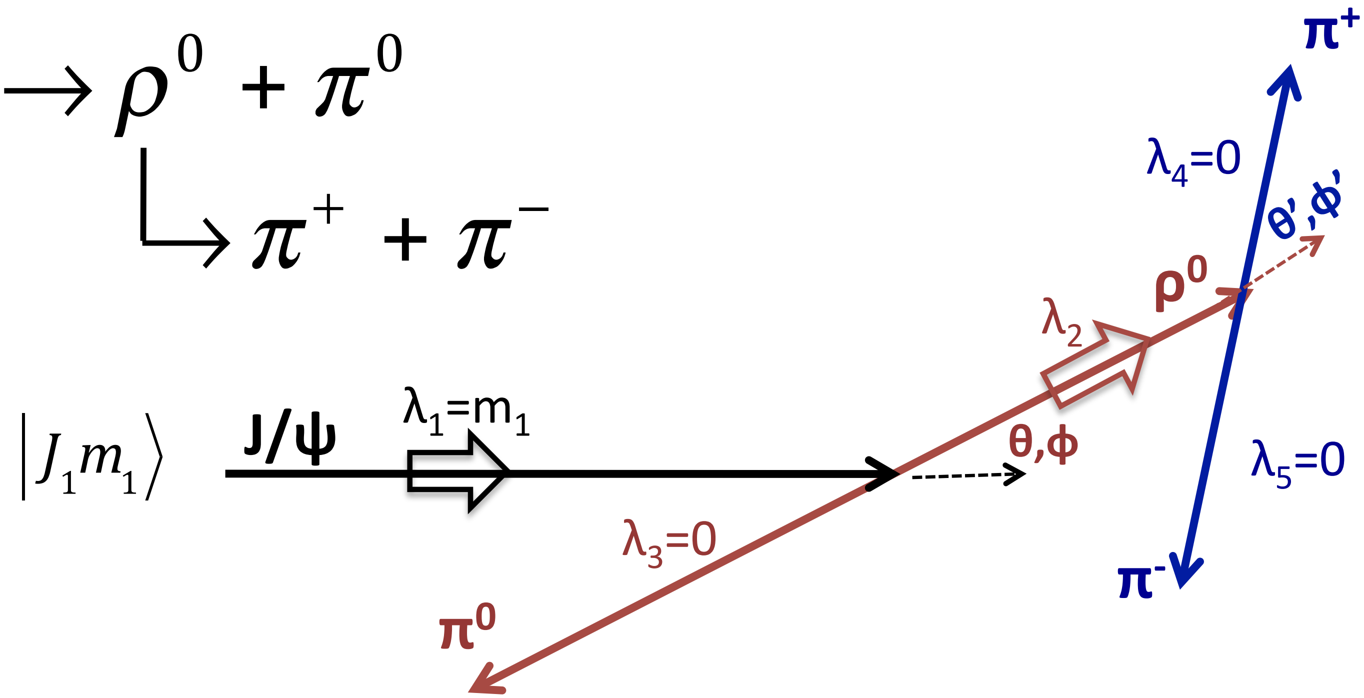
$$A(\theta,\phi,\theta',\phi',m_1,\lambda_3,\lambda_4,\lambda_5)=$$
$$\sqrt{\frac{(2J_1+1)}{4\pi}\frac{(2s_2+1)}{4\pi}}\sum_{\lambda_2} A_{\lambda_2\lambda_3} D^{S_1\,*}_{m_1\lambda_2-\lambda_3}(\theta,\phi) B_{\lambda_4\lambda_5} D^{s_2\,*}_{\lambda_2\lambda_4-\lambda_5}(\theta',\phi')$$

here processes with different values of λ_2 are coherent & interfere

Let's apply this to our simple example

add $\rho^0 \rightarrow \pi^+\pi^-$ decay

$$J/\psi \rightarrow \rho^0 + \pi^0$$
$$\hookrightarrow \pi^+ + \pi^-$$



$$A(\theta,\phi,\theta',\phi',m_1,0,0,0\ ) = \sqrt{\frac{(2J_{J/\psi}+1)}{4\pi}\frac{(2s_\rho+1)}{4\pi}}\sum_{\lambda_2=\pm1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')$$

Evaluate: $$\sum_{\lambda_2=\pm1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')$$

$A_{-10} = -A_{10}$

$D^{1*}_{1\pm1}(\theta,\phi) = e^{i\phi}\left(e^{\mp i\phi}(1\pm\cos\theta)/2\right)$

$$D^{1*}_{\pm10}(\theta',\phi') = e^{\pm i\phi'} d^{1}_{\pm10}(\theta')$$
$$= \mp\tfrac{1}{\sqrt{2}} e^{\pm i\phi'} \sin\theta'$$

Evaluate: $$\sum_{\lambda_2=\pm 1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')$$

$A_{-10} = -A_{10}$

$D^{1*}_{1\pm 1}(\theta,\phi) = e^{i\phi}\left(e^{\mp i\phi}(1 \pm \cos\theta)/2\right)$

$$D^{1*}_{\pm 10}(\theta',\phi') = e^{\pm i\phi'} d^{1}_{\pm 10}(\theta')$$
$$= \mp \tfrac{1}{\sqrt{2}} e^{\pm i\phi'} \sin\theta'$$

$$\sum_{\lambda_2=\pm 1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')$$
$$= \tfrac{-1}{2\sqrt{2}} A_{10} B_{00} e^{i\phi}\left(e^{-i\phi}(1+\cos\theta)e^{i\phi'}\sin\theta' + e^{i\phi}(1-\cos\theta)e^{-i\phi'}\sin\theta'\right)$$

Evaluate: $\displaystyle\sum_{\lambda_2=\pm1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')$

$A_{-10} = -A_{10}$

$D^{1*}_{1\pm1}(\theta,\phi) = e^{i\phi}\left(e^{\mp i\phi}(1\pm\cos\theta)/2\right)$

$D^{1*}_{\pm10}(\theta',\phi') = e^{\pm i\phi'} d^{1}_{\pm10}(\theta')$

$= \mp\frac{1}{\sqrt{2}} e^{\pm i\phi'} \sin\theta'$

$$\sum_{\lambda_2=\pm1} {''}_{\lambda_2 0} \#^{1*}_{1\lambda_2}(\theta,\phi) \$_{00} \#^{1*}_{\lambda_2 0}(\theta',\phi')$$

$$\underset{!}{=} \frac{-1}{2\sqrt{2}}\, {''}_{10} \$_{00} \%^{\&\phi}\left(\%^{-\&\phi}(1+\cos\theta)\%^{\&\phi'}\sin\theta' + \%^{\&\phi}(1-\cos\theta)\%^{-\&\phi'}\sin\theta'\right)$$

$$= \frac{-1}{2\sqrt{2}} A_{10} B_{00} e^{i\phi} \sin\theta' \left((e^{i(\phi'-\phi)} + e^{-i(\phi'-\phi)}) + \cos\theta (e^{i(\phi'-\phi)} - e^{-i(\phi'-\phi)}) \right)$$

Evaluate: $$\sum_{\lambda_2=\pm1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')$$

$A_{-10} = -A_{10}$

$D^{1*}_{1\pm1}(\theta,\phi) = e^{i\phi}\left(e^{\mp i\phi}(1\pm\cos\theta)/2\right)$

$D^{1*}_{\pm10}(\theta',\phi') = e^{\pm i\phi'} d^{1}_{\pm10}(\theta')$
$= \mp\frac{1}{\sqrt{2}} e^{\pm i\phi'} \sin\theta'$

$$\sum_{\lambda_2=\pm1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')$$

$$= \frac{-1}{2\sqrt{2}} A_{10} B_{00} e^{i\phi}\left(e^{-i\phi}(1+\cos\theta)e^{i\phi'}\sin\theta' + e^{i\phi}(1-\cos\theta)e^{-i\phi'}\sin\theta'\right)$$

$$= \frac{-1}{2\sqrt{2}} A_{10} B_{00} e^{i\phi}\sin\theta'\left((e^{i(\phi'-\phi)} + e^{-i(\phi'-\phi)}) + \cos\theta(e^{i(\phi'-\phi)} - e^{-i(\phi'-\phi)})\right)$$

$$\boxed{= \frac{-1}{\sqrt{2}} A_{10} B_{00} e^{i\phi}\sin\theta'\left(\cos(\phi'-\phi) + i\cos\theta\sin(\phi'-\phi)\right)}$$

Evaluate: $$\sum_{\lambda_2=\pm1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')$$

$A_{-10} = -A_{10}$

$D^{1*}_{1\pm1}(\theta,\phi) = e^{i\phi}\left(e^{\mp i\phi}(1\pm\cos\theta)/2\right)$

$D^{1*}_{\pm10}(\theta',\phi') = e^{\pm i\phi'} d^{1}_{\pm10}(\theta')$
$= \mp\frac{1}{\sqrt{2}} e^{\pm i\phi'}\sin\theta'$

$$\sum_{\lambda_2=\pm1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')$$

$$= \frac{-1}{2\sqrt{2}} A_{10} B_{00} e^{i\phi}\left(e^{-i\phi}(1+\cos\theta)e^{i\phi'}\sin\theta' + e^{i\phi}(1-\cos\theta)e^{-i\phi'}\sin\theta'\right)$$

$$= \frac{-1}{2\sqrt{2}} A_{10} B_{00} e^{i\phi}\sin\theta'\left((e^{i(\phi'-\phi)} + e^{-i(\phi'-\phi)}) + \cos\theta(e^{i(\phi'-\phi)} - e^{-i(\phi'-\phi)})\right)$$

$$\boxed{= \frac{-1}{\sqrt{2}} A_{10} B_{00} e^{i\phi}\sin\theta'\left(\cos(\phi'-\phi) + i\cos\theta\sin(\phi'-\phi)\right)}$$

$$\left.\frac{d\Gamma}{d(\theta,\theta',(\phi'-\phi))}\right|_{m_{J/\psi}=+1} \propto \left|\sum_{\lambda_2=\pm1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')\right|^2$$

$$= \tfrac{1}{2}\left|A_{10}B_{00}\right|^2 \sin^2\theta' \left|\cos(\phi'-\phi) + i\cos\theta\sin(\phi'-\phi)\right|^2$$

$$\left.\frac{d\Gamma}{d(\theta,\theta',(\phi'-\phi))}\right|_{m_{J/\psi}=+1} \propto \left|\sum_{\lambda_2=\pm 1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')\right|^2$$

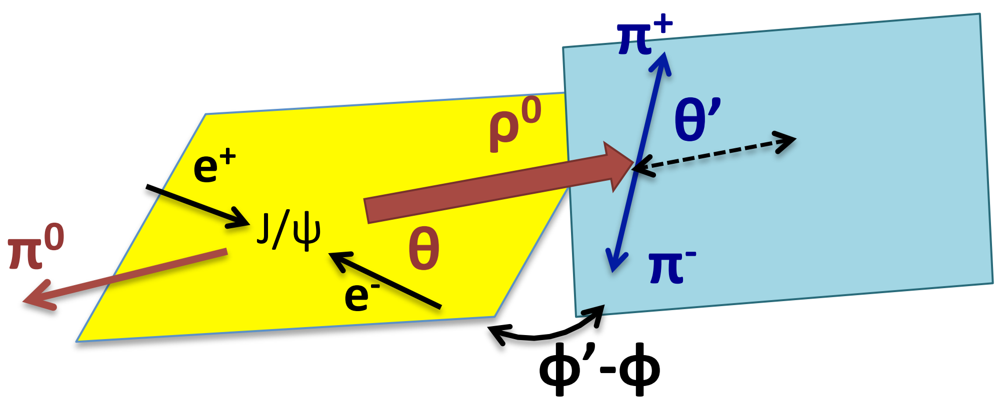
$$= \tfrac{1}{2}\left|A_{10}B_{00}\right|^2 \sin^2\theta' \left|\cos(\phi'-\phi) + i\cos\theta \sin(\phi'-\phi)\right|^2$$

$$= \tfrac{1}{2}\left|A_{10}B_{00}\right|^2 \sin^2\theta' \left(1+\cos^2\theta + \sin^2\theta \cos 2(\phi'-\phi)\right)$$

$$\left.\frac{d\Gamma}{d(\theta,\theta',(\phi'-\phi))}\right|_{m_{J/\psi}=+1} \propto \left|\sum_{\lambda_2=\pm1} A_{\lambda_2 0} D^{1*}_{1\lambda_2}(\theta,\phi) B_{00} D^{1*}_{\lambda_2 0}(\theta',\phi')\right|^2$$

$$= \tfrac{1}{2}\left|A_{10}B_{00}\right|^2 \sin^2\theta' \left|\cos(\phi'-\phi) + i\cos\theta\sin(\phi'-\phi)\right|^2$$

$$= \tfrac{1}{2}\left|A_{10}B_{00}\right|^2 \sin^2\theta' \left(1+\cos^2\theta + \sin^2\theta\cos 2(\phi'-\phi)\right)$$



4 variables that specify the event: $m_{\pi_2\pi_3}$, θ, θ', ϕ'-ϕ

References

An Angular Distribution Cookbook

Rob Kutschke

January 18, 1996

posted on the class website

CALT-68-1148
DOE RESEARCH AND
DEVELOPMENT REPORT

An Experimenter's Guide to the Helicity Formalism

Jeffrey D. Richman
California Institute of Techology
Pasadena, California 91125

M. Jacob and G. C. Wick, *Ann. Phys. (N.Y.)*, **7**, 404 (1959).

S. U. Chung, "Spin Formalisms", CERN Yellow Report 71-8, 1971.