Finite Volume Correction of the 3-body Problem in Lattice QCD

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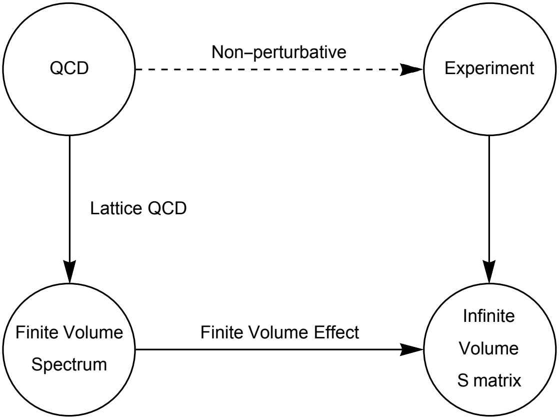
Outline

Introduction

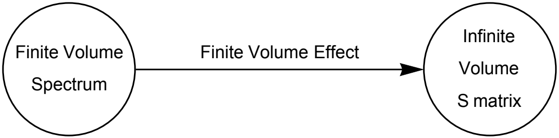
- Lattice QCD has been widely used in hadron physics and nuclear physics.
  - Hadron spectrum

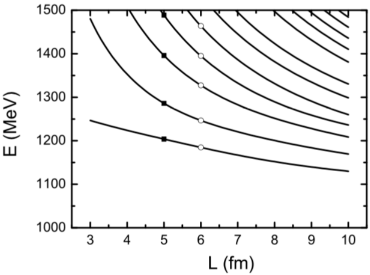


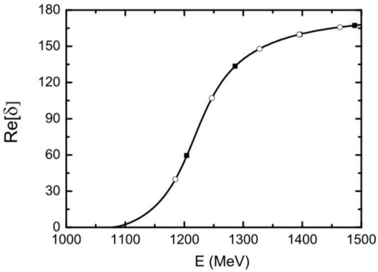
- Output of lattice QCD $\neq$ Physical measurement
  - Finite volume effect

Introduction

- Extraction of physical observable
  - Lattice simulation. 2-point function $\to E(L)$
  - Physical observables, phase shift δ, etc. $\to$ Resonance mass and width

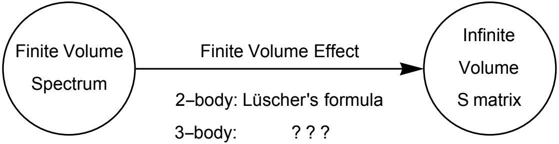






$\pi N \to \Delta \to \pi N$ (arXiv:1611.05970)

Introduction



- Finite volume correction
  - 2-body process. $\pi\pi$, πK, KN scattering
  - 3-body process
- Our aim: transparent theoretical framework, apply in $\omega \to 3\pi, K^* \to K\pi\pi$, Roper resonance etc.
  - H.-W. Hammer, J.-Y. Pang and A. Rusetsky, JHEP 1709(2017) 109; H.-W. Hammer, J.-Y. Pang and A. Rusetsky, JHEP 1710(2017) 115; M. Döring, H.-W. Hammer, M. Mai, J.-Y. Pang, A. Rusetsky and J. Wu, arXiv:1802.03362
  - On-going work: relativistic kinematics, higher order operators etc.

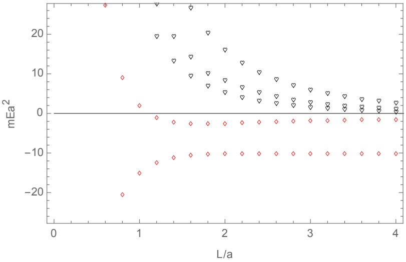
Finite Volume Effect

- Finite volume effect
  - Finite volume $\rightarrow$ discrete spectra

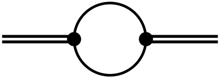
$$E = E(L) \xrightarrow{L\rightarrow\infty} \begin{cases} \text{bound state energy} \\ \text{scattering threshold} \end{cases}$$

Energy shift $\Delta E = E(L) - E(\infty)$

  - Bound state $\Delta E \sim O(e^{-L/R})$
  - Scattering state $\Delta E(L) \sim O(1/L^n)$

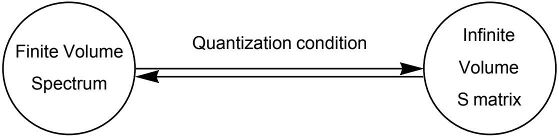


- Origin of finite volume effect
  - loop integral $\rightarrow$ loop sum



Quantization Condition

- Quantization condition is the equation which determines finite volume spectrum through physical observable.
  - Given physical observable, finite volume spectrum can be solved out
  - Inversely, given finite volume spectrum, physical observable can be extracted



- Quantization condition in scattering theory
  - Energy level $\leftrightarrow$ Pole of scattering amplitude.
  - Finite volume scattering amplitude $\mathscr{M}_L$

$$\det(\mathscr{M}_L^{-1}) = 0$$

  - Relation between finite volume scattering amplitude and infinite volume amplitude

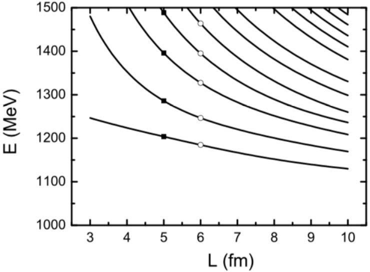
$$\mathscr{M}_L = f(\mathscr{M}, L)$$
$$\Rightarrow \quad E = E(L)$$

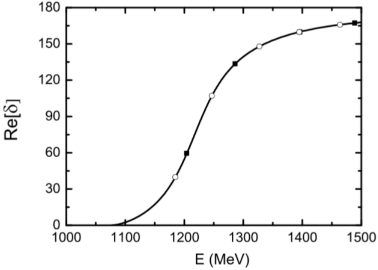
2-body Quantization Condition

- 2-body elastic process
  - Finite volume amplitude $\mathscr{M}_L^{-1} = \mathscr{M}^{-1} + i\rho + \mathscr{Z}_{00,00}(L)$
  - Infinite volume amplitude $\mathscr{M}^{-1} = \rho \cot\delta - i\rho$
  - Lüscher formula (M. Lüscher, NPB 354(1991) 531)

Lüscher formula: $\rho \cot\delta = \mathscr{Z}_{00,00}(E,L)$

  - Scattering length of $\pi\pi, \pi K, KN$ (G. Meng *et. al.* , Int. J. Mod. Phys. A19(2004) 4401; S. Beane *et. al.* , Phys. Rev. D74(2006) 114503; J. Dudek, *et. al.* , Phys. Rev. D86(2012) 034031 *etc.*)





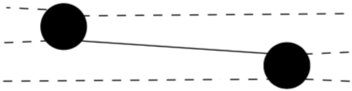
$\pi N \to \Delta \to \pi N$ (arXiv:1611.05970)

Generalization of Quantization Condition

- Coupled-channel
  - Finite volume amplitude $\mathscr{M}_{L,ij}^{-1} = K_{ij}^{-1} + \mathscr{Z}_{ij}(L)$
  - Generalized Lüscher formula (C. Liu *et al.* JHEP 0507(2005) 011; P. Guo *et al.* , Phys. Rev. D88(2013) 014501)

Quantization condition: $\det\left(K^{-1}(\delta_i, \gamma_i) + \mathscr{Z}\right) = 0$

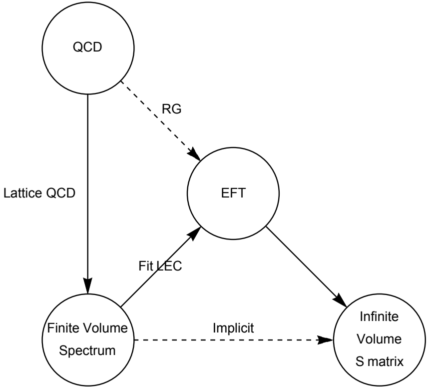
  - $\pi K - \eta K$ coupled channel (M. Lage *et al.* , Phys. Lett. B68(2009) 439, J. Dudek *et al.* , Phys. Rev. Lett. 113(2014) 182001 *etc.*)
- 3-body process
  - $3 \to 3$ scattering amplitude includes two free momenta
  - Singularity in $3 \to 3$ scattering amplitude (M. Hansen *et al.* , Phys. Rev. D90(2014) 116003)



  - The intermediate effective field theory (EFT) is necessary.

EFT in Lattice QCD

- EFT in Lattice QCD
  - LQCD generates finite volume spectrum
  - Finite volume spectrum fits LEC of EFT by quantization condition
  - EFT gives physical observable in the infinite volume
  - Related to QCD



- EFT in phenomenology
  - Fit LEC from physical observable
  - Solve out another physical observable
  - Relation with QCD is unknown

Section 2

3-body Finite Volume Effect in Effective Theory

Particle-dimer Formalism

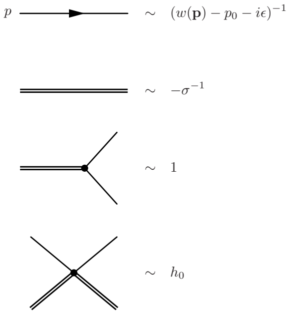
Effective field theory in particle-dimer picture

$$\mathscr{L}_{PD} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi + \sigma \left(T^\dagger T + \cdots \right) + \left(\frac{1}{2} T^\dagger \psi\psi + \cdots + \text{h.c.} \right) + h_0 T^\dagger T \psi^\dagger \psi + \cdots$$

- The simple model
  - Identical non-relativistic particle ψ
  - scalar dimer T
  - Interaction without gradient
- Extendable
  - More particle species, relativistic kinematics
  - Higher partial-wave dimer $T, T_{ij}, \cdots$
  - Operator with gradient

$$T^\dagger \psi \overleftrightarrow{\nabla}^2 \psi, T^\dagger T \psi^\dagger \overleftrightarrow{\nabla}^2 \psi$$

$(\overleftrightarrow{\nabla} = (\nabla - \overleftarrow{\nabla})/2)$



Particle-dimer Formalism

- Dimer $\neq$ approximation of 2-particle state
  - The theory made of only single particle

$$\mathscr{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi - \frac{C_0}{2} \psi^\dagger \psi^\dagger \psi \psi + \frac{C_2}{4} \left(\psi^\dagger \overleftrightarrow{\nabla}^2 \psi^\dagger \psi \psi + \text{h.c.} \right)$$
$$+ \frac{D_0}{6} \psi^\dagger \psi^\dagger \psi^\dagger \psi \psi \psi + \frac{D_2}{12} \left(\psi^\dagger \psi^\dagger \overleftrightarrow{\nabla}^2 \psi^\dagger \psi \psi \psi + \text{h.c.} \right) + \text{higher-order.}$$

  - The corresponding particle-dimer formalism

$$\mathscr{L}_{PD} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi + \sigma T^\dagger T + \left(\frac{1}{2} T^\dagger \psi \psi + f_2 T^\dagger \psi \overleftrightarrow{\nabla}^2 \psi + \text{h.c.} \right)$$
$$+ h_0 T^\dagger T \psi^\dagger \psi + h_2 T^\dagger T \psi^\dagger \overleftrightarrow{\nabla}^2 \psi + \text{higher-order.}$$

  - Integrate out dimer field

$$\int \mathscr{D}\psi \mathscr{D}\psi^\dagger e^{-S[\psi,\psi^\dagger,J]} = \int \left(\mathscr{D}\psi \mathscr{D}\psi^\dagger \mathscr{D}T \mathscr{D}T^\dagger \cdots \right) e^{-S_{PD}[\psi,\psi^\dagger,T,T^\dagger,\cdots,J]}$$

- Low-energy power counting (Dimer is heavy field)

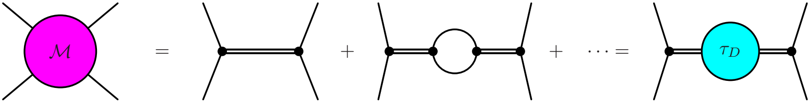
2-body Sector

- The 2-body sector for scalar dimer

$$\mathscr{L}_{PD}^{(2)} = \sigma T^\dagger T + \left(\frac{1}{2} T^\dagger \psi\psi + \text{h.c.}\right)$$

- Fix parameters Match to 2-2 on-shell rest frame amplitude determined by phase shift

$$\mathscr{M}(E) = \begin{cases} -\frac{1}{\sigma} + \frac{1}{\sigma} I \frac{1}{\sigma} - \cdots = \frac{1}{-\sigma - I(p)} \\ \frac{8\pi}{m} \frac{1}{p\cot\delta - ip} \to \frac{8\pi}{m} \frac{1}{-a^{-1} - ip} \end{cases} \hookrightarrow \text{match equation :} \quad -\sigma = -\frac{m}{8\pi a}$$



- Extract full dimer propagator

$$\mathscr{M}(\mathbf{q}; E_D) = \frac{1}{-\sigma - I(q^*)} \qquad \hookrightarrow \tau_D^{-1}(\mathbf{q}; E_D) = -a^{-1} + \sqrt{(q^*)^2 - i\varepsilon}$$

Here $q^* = \sqrt{\frac{1}{4}\mathbf{q}^2 - mE_D}$ is on-shell relative momentum.

2-body Sector (Generalized)

- The general 2-body sector for scalar dimer

$$\mathscr{L}_{PD}^{(2)} = \sigma T^\dagger T + \left(\frac{1}{2} T^\dagger \psi f(-i\nabla)\psi + \text{h.c.}\right)$$

Here $f(-i\nabla)$ is polynomial of $-i\nabla$.

- Match equation and dimer propagator

match equation : $\quad -\sigma = -\frac{m}{8\pi a} \quad \rightarrow \quad -\sigma f^{-2}(p) = -\frac{m}{8\pi} p \cot\delta(p)$

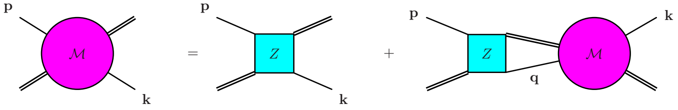
dimer propagator : $\quad \tau_D^{-1}(\mathbf{q}; E_D) = \quad q^* \cot\delta(q^*) + \sqrt{(q^*)^2 - i\varepsilon}$

- High spin dimer
- 2-body scattering amplitude as input
  - 2-body parameters fixed at the beginning
  - 3-body low energy coefficients

3-body Sector

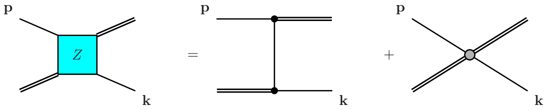
- $3 \to 3$ scattering amplitude $\Rightarrow$ Particle-dimer scattering amplitude
- Particle-dimer scattering equation

$$\mathcal{M}(\mathbf{p},\mathbf{k};E) = Z(\mathbf{p},\mathbf{k};E) + 8\pi \int^{\Lambda} \frac{d^3q}{(2\pi)^3} Z(\mathbf{p},\mathbf{q};E)\tau(\mathbf{q},E)\mathcal{M}(\mathbf{q},\mathbf{k};E)$$



1) Particle-dimer propagator $\tau^{-1}(\mathbf{q};E) = q^* \cot\delta(q^*) + \sqrt{(q^*)^2 - i\varepsilon}$, $(q^*)^2 = \frac{3}{4}\mathbf{q}^2 - mE$

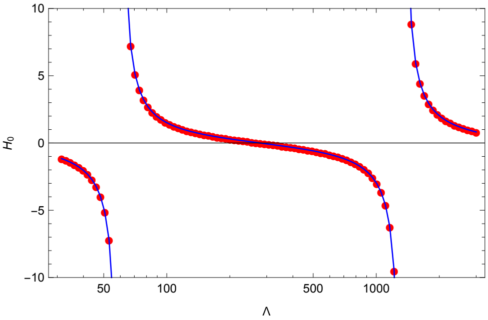
2) Particle-dimer potential



$$Z(\mathbf{p},\mathbf{q};E) = \frac{1}{\mathbf{p}^2 + \mathbf{q}^2 + \mathbf{pq} - mE} + \frac{H_0(\Lambda)}{\Lambda^2} + \frac{H_2(\Lambda)}{\Lambda^4}(\mathbf{p}^2 + \mathbf{q}^2) + \cdots$$

Physical Observable and Regularization Independence

- Solve out 3-body observable from particle-dimer scattering equation
  energy of 3-body bound state, asymptotic normalization coefficient
- Regularization
  momentum truncation by cutoff Λ in particle-dimer scattering equation
- Counter-term
  3-body force $H_0(\Lambda)/\Lambda^2$
- Physical observable: Λ-independent



1. 2-body matching $a = 1,\ r = 0, \cdots$
2. 3-body parameter

$$H_0(\Lambda) = \frac{\sin(s_0\ \text{arcsinh}\,\frac{\sqrt{3}\Lambda}{2\kappa} + \arctan s_0)}{\sin(s_0\ \text{arcsinh}\,\frac{\sqrt{3}\Lambda}{2\kappa} - \arctan s_0)}$$

$$H_2 = H_4 = \cdots = 0$$

3. Running of $H_0(\Lambda)$
   $\Rightarrow$ 3-body bound state energy is Λ-indept

Subsection 2

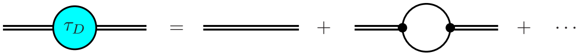
Finite Volume Correction

Finite Volume Correction

- Infinite volume $\rightarrow$ Finite volume cubic box
- Momentum: continuous $\rightarrow$ discrete $(\mathbf{p} = 2\pi\mathbf{n}/L)$
- Finite volume propagator

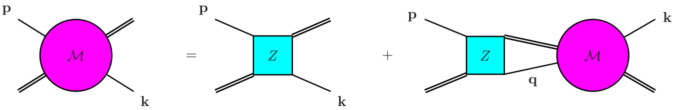
$$\tau_L^{-1}(\mathbf{q};E) = q^* \cot\delta(q^*) + \mathrm{Re}\sqrt{-q^{*2}} - S_L(\mathbf{q};E)$$

$$S_L(\mathbf{q};E) = 4\pi \sum\!\!\!\!\!\!\!\int_{\mathbf{l}} \frac{1}{\mathbf{q}^2 + \mathbf{l}^2 + \mathbf{ql} - mE}$$



- Particle-dimer scattering equation in a finite volume

$$\mathcal{M}_L(\mathbf{p},\mathbf{k};E) = Z(\mathbf{p},\mathbf{k};E) + \frac{8\pi}{L^3}\sum_{\mathbf{q}}^{\Lambda} Z(\mathbf{p},\mathbf{q};E)\tau_L(\mathbf{q};E)\mathcal{M}_L(\mathbf{q},\mathbf{k};E)$$



Quantization Condition

- 3-body quantization condition

$$\det(\mathscr{M}_L^{-1}) = 0 \Rightarrow \det(\tau_L^{-1} - \frac{8\pi}{L^3} Z) = 0$$

  - 3-body finite volume spectrum is determined by the pole of $\mathscr{M}_L$
  - Quantization condition dependes on low energy coefficients (LEC)

- Symmetry in a finite volume cubic box
  - $SO(3)$ group $\rightarrow \mathscr{M}(\mathbf{p},\mathbf{q}) = \mathscr{M}(R\mathbf{p}, R\mathbf{q}) \rightarrow$ Partial wave expansion (PWE)
  - Finite volume cubic box: O_h group $\rightarrow \mathscr{M}_L(\mathbf{p},\mathbf{q}) = \mathscr{M}_L(R_O\mathbf{p}, R_O\mathbf{q})$ $(R_O \in O_h)$
  - O_h: 48 group elements & 10 irreps. $\Gamma = A_1^{\pm}, A_2^{\pm}, E^{\pm}, T_1^{\pm}, T_2^{\pm}$.
  - PWE breaks down $\rightarrow$ Discrete PWE

$$f(\mathbf{p}) = f(g\mathbf{p}_0(s)) = \sum_{\Gamma,ij} f_{ij}^{(\Gamma)}(s) T_{ij}^{(\Gamma)}(g).$$

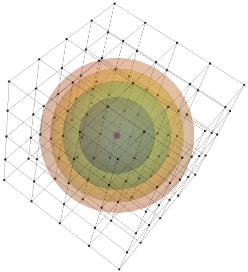
T is matrix of irrep Γ.

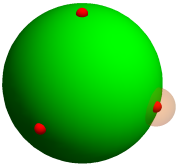
Shell Structure

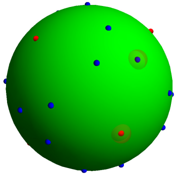
- Shell structure

  Shell is a set of momenta with the same $|\mathbf{p}|$, which can be obtained from reference momentum $\mathbf{p}_0$, $\mathbf{p} = g\mathbf{p}_0, \quad g \in O_h$.

  The momenta unrelated by the O_h, but having $|\mathbf{p}| = |\mathbf{p}'|$, belong to the different shells.







- Sum over shells All momenta in a given shell are produced from reference momentum.

$$\sum_{\mathbf{p}} f(\mathbf{p}) = \underbrace{\sum_{s}}_{\text{different shells}} \underbrace{\frac{\vartheta_s}{G} \sum_{g}}_{\text{orientations inside shell } s} f(g\mathbf{p}_0(s)).$$

Reduction of the Quantization Condition

Homogeneous STM equation in a finite volume, $\mathscr{F}(\mathbf{p}) = \frac{8\pi}{L^3}\sum_{\mathbf{q}}^{\Lambda} Z(\mathbf{p},\mathbf{q};E)\tau_L(\mathbf{q};E)\mathscr{F}(\mathbf{q})$.

- Expansion of $\mathscr{F}(\mathbf{p})$
  $\mathscr{F}(\mathbf{p}) = \mathscr{F}(g\mathbf{p}_0(s)) = \sum_{\Gamma,ij} \mathscr{F}_{ij}^{(\Gamma)}(s) T_{ji}^{(\Gamma)}(g) \quad \& \quad \mathscr{F}_{ij}^{(\Gamma)}(s) = \frac{s_\Gamma}{G}\sum_g T_{ji}^{(\Gamma)*}(g)\mathscr{F}(g\mathbf{p}_0(s))$.
- Propagator τ_L $\tau_L(\mathbf{q};E) = \tau(g\mathbf{q};E)$.
  $\tau_L(\mathbf{q};E) = \tau_L(g\mathbf{q}_0(r);E) = \frac{1}{\vartheta_r}\tau_L(r;E)$.
- Expansion of Z $Z(\mathbf{p},\mathbf{q};E) = Z(g\mathbf{p},g\mathbf{q};E)$.
  $Z(\mathbf{p},\mathbf{q};E) = Z(g\mathbf{p}_0(s), g'\mathbf{q}_0(r);E) = \sum_{\Gamma,ij,n} s_\Gamma T_{ij}^{(\Gamma)}(g) Z_{jn}^{(\Gamma)}(s,r;E) T_{in}^{(\Gamma)*}(g')$.
  $Z_{jn}^{(\Gamma)}(s,r;E) = \frac{1}{G}\sum_g T_{nj}^{(\Gamma)*}(g) Z(\mathbf{p}_0(s), g\mathbf{q}_0(r);E)$.

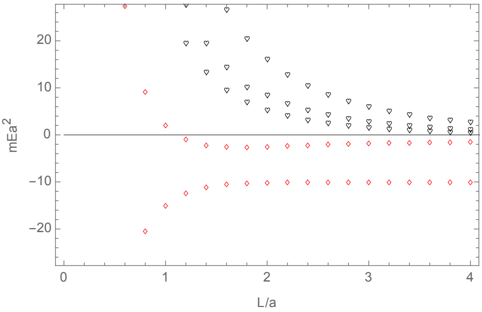
$$\mathscr{F}_{ij}^{(\Gamma)}(s) = \frac{8\pi}{L^3}\sum_r \frac{\vartheta_r}{G}\sum_n Z_{jn}^{(\Gamma)}(s,r;E)\tau_L(r;E)\mathscr{F}_{ni}^{(\Gamma)}(r) \Rightarrow$$

$$\boxed{\det\left(\tau^{-1}(r;E)\delta_{sr}\delta_{ij} - \frac{8\pi}{L^3} Z_{ij}^{(\Gamma)}(s,r;E)\right) = 0}$$

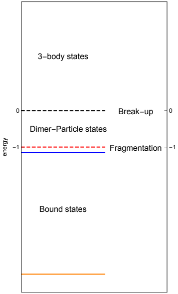
Section 3

3-body System in a Finite Volume

Finite Volume Spectrum



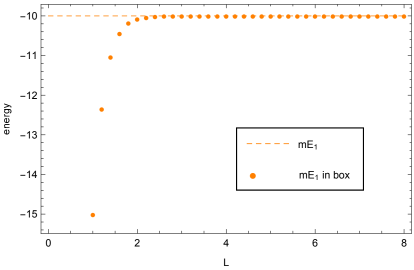
- Spectra in a finite volume cubic box
  - Particle-dimer scattering states
  - 3-body scattering states
  - Particle-dimer bound state
  - Bound state



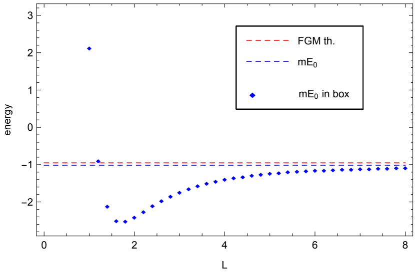
- Infinite volume spectrum
  - Particle-dimer threshold
  - 3-body threshold
  - Bound states

Bound States

Deep one



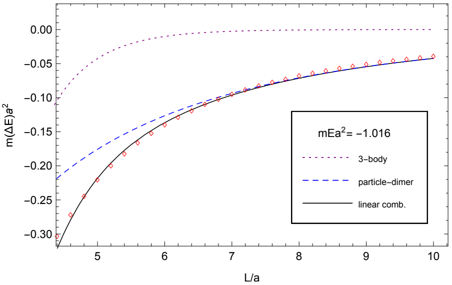
Shallow one

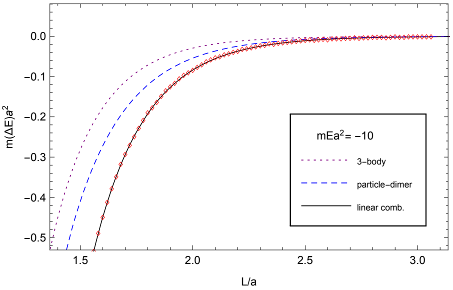


$$\Delta E = \frac{\kappa^2}{m}\left[\frac{1}{(\kappa L)^{3/2}} C \exp\left(-\frac{2}{\sqrt{3}}\kappa L\right) + \frac{1}{\sqrt{(\kappa a)^2 - 1}}\frac{1}{(\kappa L)} C' \exp\left(-\frac{2}{\sqrt{3}}\sqrt{\kappa^2 - a^{-2}} L\right)\right]$$

- 3-body contribution: $\frac{1}{(\kappa L)^{3/2}} \exp\left(-\frac{2}{\sqrt{3}}\kappa L\right)$ (M. Lüscher, NPB 354 (1991) 531)
- Particle-dimer contribution: $\frac{1}{\sqrt{(\kappa a)^2-1}}\frac{1}{(\kappa L)} \exp\left(-\frac{2}{\sqrt{3}}\sqrt{\kappa^2 - a^{-2}} L\right)$ (U. Meißner et. al. , PRL 114(9) (2015), 091602)

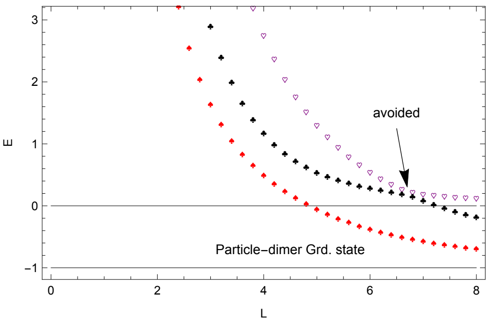
Identification of Bound States





- Shallow one $mE_0 = -1.016$
  - Dominated by particle-dimer contribution $\to$ predominately particle-dimer state.
- Deep one $mE_1 = -10$
  - Particle-dimer and 3-body contributions are comparable in magnitude $\to$ mixture.

Scattering States



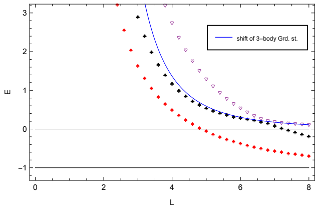
- Particle-dimer ground state
- Avoided level crossing

- Shift of 3-body ground state

$$mE(L) = \frac{12\pi a}{L^3} - \frac{12a^2}{L^4}\mathscr{I} + \frac{12a^3}{\pi L^5}\left(\mathscr{I}^2 + \mathscr{J}\right) + o(\frac{1}{L^5}).$$

(S. Beane *et.al.*, arXiv:0707.1670, S. Sharpe, arXiv:1707.04279)

- 3-body $\leftrightarrow$ particle-dimer



Conclusions

- The non-relativistic quantization condition of 3-body problem is obtained.
- In a finite volume, the quantization condition is projected onto the different irreps of the octahedral group.
- The finite volume spectrum is obtained and identified.

- Outlook
  - Generalize the quantization condition to Lorentz invariant form
  - Use the method to predict the outcome of lattice simulation in the realistic systems.

Thank you for your attention!