Where do the usual Coulomb correction factor come from

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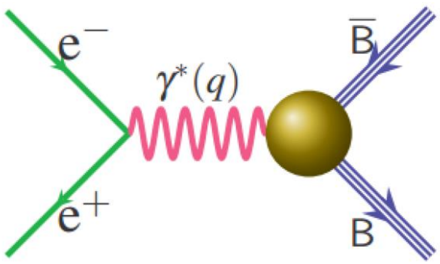
Outline

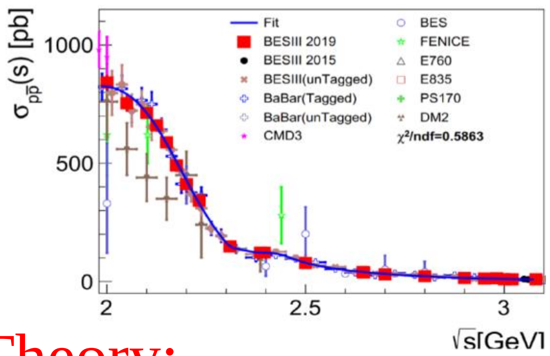
- ☞ **Motivations**
- ☞ **Two-body wave function in Coulomb potential**
- ☞ **Threshold effects by quantum mechanics**
- ☞ **Summary**

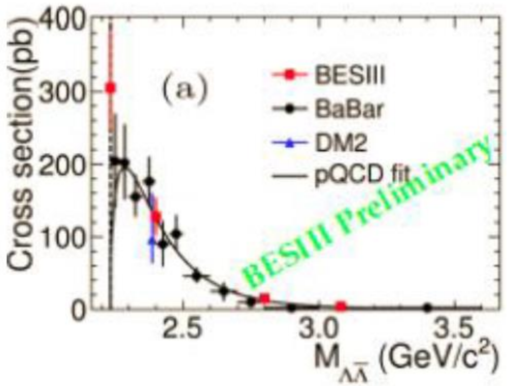
Motivations

Experiments: $e^+e^- \to \gamma^* \to B\bar{B}$



- $e^+e^- \to p\bar{p}$: an enhancement and wide plateau
- $e^+e^- \to \Lambda\bar{\Lambda}$: non-zero cross section near threshold





Theory:

- Born cross section for the process

$$\sigma = \frac{4\pi\alpha^2\beta}{3s} C[|G_M(s)|^2 + \frac{2M^2}{s}|G_E(s)|^2]$$

- Baryon speed $\beta = \sqrt{1 - 4M^2/s}$

- Coulomb factor C , non-zero cross section at threshold $\beta \to 0$

$$C = \frac{y}{1 - exp(-y)} \qquad y = \pi\alpha M/\beta\sqrt{s} \qquad C \to= \frac{\pi\alpha M}{\sqrt{s}}\frac{1}{\beta} \quad (\beta \to 0)$$

- Where the factor C come from? How to understand C?
- Is it correct or reliable for baryons? Does C describes threshold effect truly?

Principles and definition

- Reference

A.D. Sakharov, Sov. Phys. Usp. 34(**5**), 1991
Interaction of the electron and positron in pair production

- Hamiltonian and eigenfunction

$$H = H_0 + H_I \qquad H_0 = \tilde{H}_0 + W$$

$$\tilde{H}_0 \tilde{\Psi} = E_0 \tilde{\Psi}$$

- Matrix

Representing from initial state to final state $i \to f$:
Absent interaction: $\tilde{V}$
Existing interaction: V

- Correction factor of interaction

Differential probability for process $i \to f$
Absent interaction: $d\tilde{\omega}$
Existing interaction: $d\omega$
Correction factor :

$$T = \frac{d\omega}{d\tilde{\omega}} = |\frac{V}{\tilde{V}}|^2$$

Formulas

A.D. Sakharov, Sov. Phys. Usp. 34(**5**), 1991

Kinematics
Laboratory: p_+, p_-
Center of mass: k_+, k_-

- Transition matrix elements for $p_+p_- \longrightarrow p_+' p_-'$

$$V_{p'_+p'_-} = \int \Psi_0^* H_I \Psi_{p'_+p'_-} dq$$

- Interaction wave function

Linear combination of zero-level wave function (in Lab.)

$$\Psi_{p'_+p'_-} = \int d^3p_+ d^2p_- < p_+p_-|c|p'_+p'_- > \tilde{\Psi}_{p_+p_-}$$

c is some unitary singular matrix very close to δ matrix

- Interaction transition matrix element

$$V_{p'_+p'_-} = \int d^3p_+ d^2p_- < p_+p_-|c|p'_+p'_- > \tilde{V}_{p_+p_-}$$

- Perturbative property

α=1/137, very small, the final state is very close to the initial state

$$c_p \to \tilde{c} = \delta(p_+ - p'_+)\delta(p_- - p'_-) \quad \text{in the limiting case} \quad e \to 0$$

$$V_{p'_+p'_-} \approx \tilde{V}_{p_+p_-} \int d^3p_+ d^2p_- < p_+p_-|c|p'_+p'_- >$$

- Amplitude of interaction correction

$$J = \frac{V}{\tilde{V}} = \int d^3p_+ d^3p_- \cdot < p_+p_-|c|p'_+p'_- > = \int d^3p_+ d^3p_- \cdot c_p$$

Formulas

A.D. Sakharov, Sov. Phys. Usp. 34(**5**), 1991

- Matrix elements between Laboratory and center-f-mass systems

$$\underbrace{< k_+k_-|c_k|k'_+k'_- >}_{\text{Theory calculation (momentum space)}} = \sqrt{\lambda\lambda'} \underbrace{< p_+p_-|c_p|p'_+p'_- >}_{\text{Experimental measurement (momentum space)}}$$

λ, λ': Jacobians
λ≈λ'

$$J = \int d^3p_+d^3p_- \cdot c_p = \int d^3p_+d^3p_- \cdot J^{-1}c_k = \int d^3k_+d^3k_- \cdot c_k$$

- Fourier transformation
  - The relation between matrix c and wave function

$$\int d^3k_+d^3k_-c_k \overset{?}{=} (2\pi)^3 < 00|c_x|k'_+k'_- >$$

  - Solution of Schrodinger equation

$$-\frac{1}{2}(\Delta_+ + \Delta_-)c_x - \frac{e^2}{r}c_x = \frac{1}{2}(k'^2_+ + k'^2_-)c_x$$

$$c_x = \frac{F(i\varepsilon, 1, i[(\vec{k}'_+ - \vec{k}'_-)\cdot(\vec{x}_+ - \vec{x}_-) + |\vec{k}'_+ - \vec{k}'_-||\vec{x}_+ - \vec{x}_-|])}{|F(i\varepsilon, 1, i\infty)|}\tilde{c}_x$$

$$\boxed{\varepsilon = \frac{e^2}{v}}$$

$$\tilde{c}_x = \frac{1}{(2\pi)^3}\exp[i(k'_+x_+ + k'_-x_-)]$$

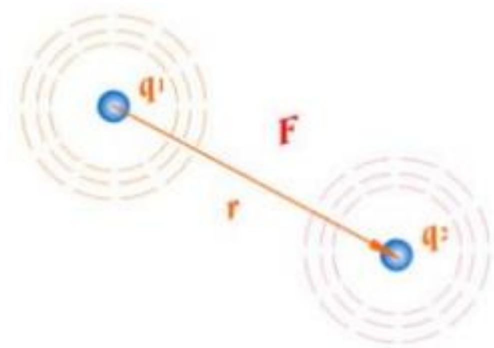
- Coulomb interaction correction factor

$$J = |F(i\varepsilon, 1, i\infty)|^{-1} = \sqrt{\frac{2\pi\varepsilon}{1-\exp(-2\pi\varepsilon)}} \qquad F_c = T = \frac{2\pi\varepsilon}{1-\exp(-2\pi\varepsilon)}$$

Two-body system in Coulomb potential

- Interaction: Coulomb potential of point charges

$$U(r) = \pm\frac{\alpha}{r}$$



- Close to threshold, nonrelativistic

$$\beta \to 0$$

- Schrodinger's equation

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

L. D. Landau, E. M .Lifshitz
Quantum Mechanics, Non-relativistic theory

- Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m_1}\Delta_1 - \frac{\hbar^2}{2m_2}\Delta_2 + U(r) \qquad \hat{H} = -\frac{\hbar^2}{2M}\Delta_R - \frac{\hbar^2}{2m}\Delta + U(r)$$

- Wave function

$$\Psi = \phi(\vec{R})\psi(\vec{r})$$

Relative motion in potential U(r), physics interesting

Motion of the center of mass, as free particle, physics ordinary

- Schrodinger equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) - \frac{1}{\hbar^2}\frac{1}{r^2}\hat{L}^2\psi + \frac{2m}{\hbar^2}[E - U(r)]\psi = 0 \qquad \psi(\vec{r}) = R(r)Y_{lm}(\theta,\phi)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) - \frac{l(l+1)}{r^2}R + \frac{2m}{\hbar^2}[E - U(r)]R = 0 \qquad \hat{L}^2 Y_{lm} = \hbar^2 l(l+1)Y_{lm}$$

Units: $\mathrm{mass:}\ m, \quad \mathrm{length:}\ \hbar^2/a\alpha, \quad \mathrm{time:}\ \hbar^3/m\alpha^2 \quad \to \quad \mathrm{energy:}\ m\alpha^2/\hbar^2$

Solutions of Schrodinger equation

- Discrete spectrum: E < 0, combined state

$$n = \frac{1}{\sqrt{-2E}} \quad E = -\frac{1}{n^2}, \quad n = 1, 2, \cdots$$

(No interesting here)

- Continuous spectrum: E > 0

$$n = \frac{-i}{\sqrt{2E}} = \frac{-i}{k} \quad \rho = 2ikr$$

$$\boxed{p = mv = \hbar k \quad p = \hbar k \quad v = \frac{\hbar}{m} k}$$

- Solutions: wave functions

$k/2\pi$ scale
$$R_{kl}(r) = \frac{C_{kl}}{(2l+1)!} e^{-ikr} (2kr)^l F(\frac{i}{k} + l + 1, 2l + 2, i2kr)$$

energy scale
$$R_{El}(r) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{dk}{dE}} R_{kl}(r) = \frac{1}{\hbar} \sqrt{\frac{m}{2\pi k}} R_{kl}(r)$$

$$\int \Psi_f^*(q') \Psi_f(q) dq = \delta(q' - q)$$

$$\int \Psi^*_{\phi(f')}(q') \Psi_{\phi(f)}(q) dq = \delta[\phi(f') - \phi(f)]$$

$$\Psi_{\phi(f)} = \frac{\Psi_f}{\sqrt{|d\phi(f)/df|}}$$

At origin $r = 0$, only $l = 0$ partial wave non-zero

- Renormalization constant

$$U(r) = \begin{cases} -\frac{\alpha}{r} & \text{(attractive)} \\ +\frac{\alpha}{r} & \text{(repulsive)} \end{cases}$$

- attractive potential

$$C_{kl} = 2ke^{\pi/2k} |\Gamma(l + 1 - i/k)| = \frac{\sqrt{8\pi k}}{\sqrt{1 - e^{-2\pi/k}}} \prod_{j=0}^{l} (j^2 + \frac{1}{k^2})$$

- repulsive potential

$$C_{kl} = 2ke^{\pi/2k} |\Gamma(l + 1 + i/k)| = \frac{\sqrt{8\pi k}}{\sqrt{e^{2\pi/k} - 1}} \prod_{j=0}^{l} (j^2 + \frac{1}{k^2})$$

Threshold effect?

- At origin $r = 0$

$$|R_{E0}(0)|^2 = \frac{4m}{\hbar}\frac{1}{1-\exp(-2\pi/k)} \quad (U(r) = -\frac{\alpha}{r}) \qquad k \to 0, \quad |R_{E0}(0)|^2 \to \frac{4m}{\hbar} \quad \text{finit}$$

$$|R_{E0}(0)|^2 = \frac{4m}{\hbar}\frac{1}{\exp(+2\pi/k)-1} \quad (U(r) = +\frac{\alpha}{r}) \qquad k \to 0, \quad |R_{E0}(0)|^2 \to 0 \quad \text{zero}$$

$$|R_{E0}(0)|^2 \sim F_c \quad ? \qquad |R_{E0}(0)|^2\sigma_0(s_{th}) \to 0$$

zero-point wave function can not account for threshold effect

- Consider incident beam?

L.I. Schiff, Quantum Mechanics, p141

$$I = |\vec{v}| = \sqrt{\frac{\hbar k}{m}} \quad \text{current density}$$

$$|R^I_{E0}(0)|^2 = \frac{|R_{E0}(0)|^2}{I} = \frac{m^2}{\hbar^2}\frac{1/\hbar k}{1-\exp(-2\pi/k)} = \frac{m^2}{\hbar^2}\frac{1/mv}{1-\exp(-2\pi\hbar/mv)} \quad (U(r) = -\frac{\alpha}{r})$$

Threshold effect

$$|R^I_{E0}(0)|^2 = \frac{|R_{E0}(0)|^2}{I} = \frac{m^2}{\hbar^2}\frac{1/\hbar k}{\exp(+2\pi/k)-1} = \frac{m^2}{\hbar^2}\frac{1/mv}{\exp(+2\pi\hbar/mv)-1} \quad (U(r) = +\frac{\alpha}{r})$$

- But cross section has considered incident beam

$$d\sigma_{fi} = \frac{dP_{fi}}{I} \qquad \sigma_{fi} = \frac{P_{fi}}{I}$$

Initial one particle wave function

$$\psi \sim \frac{1}{v}\exp[-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{r})]$$

- Double counting considered incident beam?

$$\tilde{\sigma}_0(s_{th}) = |\tilde{R}^I_{E0}(0)|^2 \cdot \sigma_0(s_{th}) \propto \frac{1}{I^2}$$

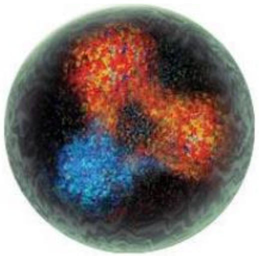
Point or nonpoint particles?

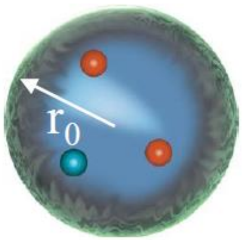
- Electron, point particle $r_{min} = 0$

  - above calculations and conclusion seems reasonable
  - at origin $r = 0$, only $l = 0$ partial wave (s wave) has non-zero contribution

$$R_{kl}(r) \;=\; \frac{C_{kl}}{(2l+1)!} e^{-ikr}(2kr)^l F(\frac{i}{k}+l+1, 2l+2, i2kr)$$

- Baryon, non-point particle, with component, $r_{min} = r_0 > 0$





  - all l partial wave contribution are allowed, l = 0,1,2……

? $$|R^I_{kl}(r_0)|2 = \frac{8\pi/k}{1-\exp(-2\pi/k)}(2r_0k)^{2l}[\prod_{j=0}^{l}(j^2+\frac{1}{k^2})]^2|F(\frac{i}{k}+l+1, 2l+1, 2ikr_0)|^2$$

  - $r \geq r_0$, diverge when $k \to 0$

  - application ? interpretation?

  - simple quantum mechanics formula still available?

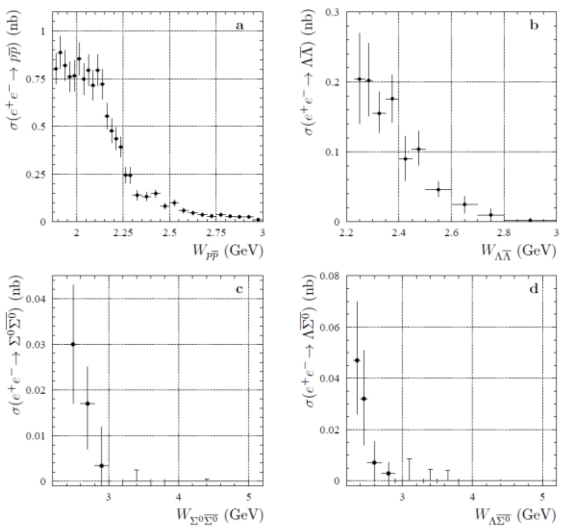
Threshold effect of other baryons final states

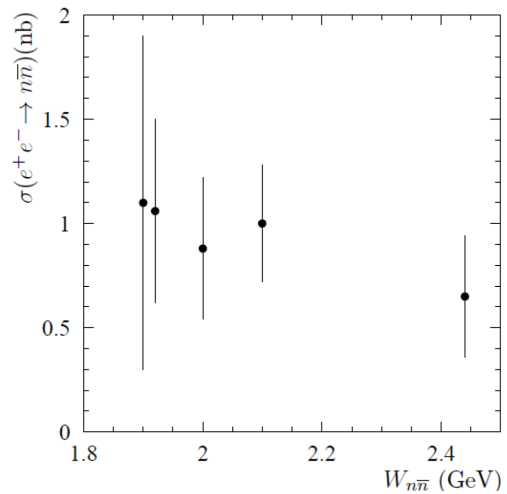
arXiv:0711.1725v3 [hep-ph] 1 Apr 2008

Unexpected features of $e^+e^- \to p\bar{p}$ and $e^+e^- \to \Lambda\bar{\Lambda}$ cross sections near threshold

Rinaldo Baldini[a,b], Simone Pacetti[a,b], Adriano Zallo[b], and Antonino Zichichi[a,d,e]

- Threshold effects seem be universal





$$C(W_{\mathcal{B}\bar{\mathcal{B}}}) = \begin{cases} 1 & \text{for neutral } \mathcal{B} \\ \dfrac{\pi\alpha/\beta}{1 - e^{-\pi\alpha/\beta}} & \text{for charged } \mathcal{B} \end{cases}$$

- Threshold effects for neutron channels

- Coulomb correction leak?
- The unexpected features of baryon pair production?
- Near threshold can be interpreted as: $\boldsymbol{B\bar{B}}$ bound states?
- Unobserved meson resonances, an attractive Coulomb interaction?
- Strong interaction on the level of constituent quark level?
- ……

X(1835), X(pp), X(nn)?

Final state of $e^+e^- \to \tau^+\tau^-$

arXiv:hep-ph/9312358v1 30 Dec 1993

UMN-TH-1232/93
TPI-MINN-93/61-T
December 1993

$e^+ e^- \to \tau^+ \tau^-$ at the threshold and beyond

Brian H. Smith   M.B. Voloshin

- Born cross section

$$\sigma_0(e^+ e^- \to \tau^+ \tau^-) = \frac{2\pi\,\alpha^2}{3s}\, v\,(3 - v^2) \qquad v = \sqrt{1 - 4m_\tau^2/s}$$

- Interpolating cross section of final state interaction

$$\bar{\sigma}(e^+e^- \to \tau^+\tau^-) = \sigma_0\,(1+h)\,F_c\left(1 + \frac{\alpha}{\pi}S(v) - \frac{\pi\,\alpha}{2v}\right)$$

- Coulomb correction $F_c = \dfrac{\pi\alpha/v}{1 - \exp(-\pi\alpha/v)}$

- Wave function correction

$$\psi_c(r) = C\,e^{-ipr}\,{}_1F_1(1 + i\lambda, 2, 2ipr)$$

$$|\psi_c(0) + \delta\psi(0)|^2 = |\psi_c(0)|^2\,(1+h)$$

$$h = \frac{2\alpha}{3\pi}\left[-2\lambda\,\mathrm{Im}\int_0^\infty dt\int_1^\infty dx\left(\frac{1+t}{t}\right)^{i\lambda}\frac{(t + z\,x\,v^{-1})^{i\lambda-1}}{(t+1+z\,x\,v^{-1})^{i\lambda+1}}\left(1 + \frac{1}{2x^2}\right)\frac{\sqrt{x^2-1}}{x^2}\right]$$

- higher correction   J. Schwinger, Particle, Sorces, and Field, Volume II

$$S(v) = \frac{1}{v}\left\{(1+v^2)\left[\frac{\pi^2}{6} + \ln\left(\frac{1+v}{2}\right)\ln\left(\frac{1+v}{1-v}\right) + 2\,\mathrm{Li}_2\left(\frac{1-v}{1+v}\right) + 2\,\mathrm{Li}_2\left(\frac{1+v}{2}\right) - 2\,\mathrm{Li}_2\left(\frac{1-v}{2}\right) - 4\,\mathrm{Li}_2(v) + \mathrm{Li}_2(v^2)\right] + \left[\frac{11}{8}(1+v^2) - 3v + \frac{1}{2}\frac{v^4}{(3-v^2)}\right]\ln\left(\frac{1+v}{1-v}\right) + 6v\,\ln\left(\frac{1+v}{2}\right) - 4v\,\ln v + \frac{3}{4}v\frac{(5-3v^2)}{(3-v^2)}\right\}$$

- Dose $e^+e^- \to$ baryons has similar correction?
  - With QCD modified form   - Need more study

Summary

- Coulomb interaction in quantum mechanics seems can interpret the threshold effect of charged point-like baryon-pair production cross section.
- Coulomb interaction formula become invalid for interpreting the threshold effect of nonpoint-like particles production cross section.
- Baryons are complex particle with inner components.
- The threshold of neutron pair cross section need new mechanism.
- What is true hadronic threshold effects mechanisms are far from clear.
- More studies for threshold effect correction are needed.